COMPRESSED SENSING USING THE DFT

1. Basic Idea of Compressed Sensing
2. Compressed Sensing Using the DFT
3. Deterministic Form of MUSIC
4. Sparsifying Transform Using the DFT
5. Tiny Example

BASIC IDEA OF COMPRESSED SENSING [1/3]

Given: M observations $y{=}Ax$ where:
x is an unknown N-vector to be computed;
y is a known M-vector which is measured;
A is a known and specified $M \times N$ matrix;
And: $M << N$ so $y{=}Ax$ is *underdetermined*.
Goal: Reconstruct x from y.

BASIC IDEA OF COMPRESSED SENSING [2/3]

How? $y{=}Ax$ is underdetermined! **But:**
Given: Sparsifying transform $z{=}Hx$ where:
H is a known $N \times N$ invertible matrix;
z is an unknown N-vector which is *sparse*:
Means: Only K of the N elements of z are non-zero,
at unknown locations (indices) $\{n_1 \dots n_K\}$.

BASIC IDEA OF COMPRESSED SENSING [3/3]

Now: Solve $y{=}Ax{=}(AH^{-1})z$. Then: $x{=}H^{-1}z$.
Where: z is sparse (most elements are zero).
How? *Basis Pursuit*: Find z minimizing the
ℓ_1 norm: $||z||_1{=}\sum_{n=1}^{N}|z_n|$ using linear programming.
Also: Other approaches (matching pursuit, etc.)
Simpler: Use DFT for compressed sensing (next).

COMPRESSED SENSING USING THE DFT

DFT: $X_k{=}\sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N}, k{=}0 \dots N{-}1$.
Given: $\{X_0, X_1 \dots X_K\}$ and $X_{N-k}{=}X_k^*$.
Goal: Reconstruct x_n from these X_k using:

Sparsifying: $z_n{=}x_n \circledcirc h_n \Leftrightarrow Z_k{=}X_k H_k$ for some h_n.
Do after show how to reconstruct z_n from Z_k.

DETERMINISTIC FORM OF MUSIC [1/2]

Given: $\{Z_0, Z_1 \dots Z_K\}$ and $Z_{N-k}{=}Z_k^*$.
Goal: Reconstruct sparse z_n: $z_n{=}0$ for $n \notin \{n_1 \dots n_K\}$.

Def: Annihilating or indicator function s_n using:

$$\begin{cases} s_n = 0 & \text{if } z_n \neq 0 \\ s_n \neq 0 & \text{if } z_n = 0 \end{cases} \qquad \begin{cases} S_k \neq 0 & 0 \leq k \leq K \\ S_k = 0 & \text{otherwise} \end{cases}$$

Then: $s_n z_n{=}0 \rightarrow S_k \circledcirc Z_k{=}0$ (cyclic convolution).

DETERMINISTIC FORM OF MUSIC [2/2]

Then: Arrange into Toeplitz system of equations

$$\begin{bmatrix} Z_0 & Z_1 & \cdots & Z_K \\ Z_1^* & Z_0 & \cdots & Z_{K-1} \\ \ddots & \ddots & \ddots & \ddots \\ Z_K^* & Z_{K-1}^* & \cdots & Z_0 \end{bmatrix} \begin{bmatrix} S_K \\ S_{K-1} \\ \vdots \\ S_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So: Right null vector of this Toeplitz matrix
Are: values of the DFT S_k of the function s_n.
Then: Zero values of s_n are locations n_i of nonzero z_n.

SPARSIFYING TRANSFORM h_n [1/2]

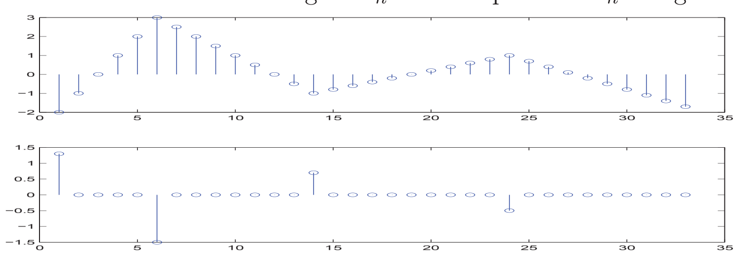
Given: x_n is known to be *piecewise linear* function.
Then: $z_n = x_{n+1} + x_{n-1} - 2x_n$, all indices mod$(N)$.
So: $h_n = \{1, \underline{-2}, 1\}$ is a sparsifying transform.

$$H(e^{j\omega}) = \text{DTFT}[h_n] = e^{j\omega} + e^{-j\omega} - 2 = 2\cos(\omega) - 2$$
$$H(e^{j\omega}) = 2(1 - \omega^2/2! + \ldots) - 2 \simeq -\omega^2 \text{ for } \omega << 1$$

Is: Frequency response of $z(t) = \frac{d^2x}{dt^2}$ zeros lines.

SPARSIFYING TRANSFORM h_n [2/2]

Piecewise linear signal x_n and its sparsified z_n using above h_n:



OVERALL PROCEDURE

Given: $\{X_0, X_1 \ldots X_K\}$ and some sparsifying h_n.
Compute: $Z_k = X_k H_k, |k| \leq K$ from H_k=DFT of h_n.
Compute: Null vector $[S_K \ldots S_0]^T$ of Toeplitz matrix.
Compute: The inverse N-point DFT s_n of $[S_0 \ldots S_K]$.
Zero values of s_n occur at $n = \{n_1 \ldots n_K\}$.
Solve: $\sum_{i=1}^{K} z_{n_i} e^{-j2\pi n_i k/N} = X_k, k = 0 \ldots K$ for z_{n_i}.
Compute: Z_k from z_{n_i} (other z_n=0); $X_k = \frac{Z_k}{H_k}$ for $|k| > K$

TINY EXAMPLE [1/2]

Given: $\{X_0 \ldots X_4\}$; sparsifying $h_n = \{1, \underline{-2}, 1\}$.
Goal: Reconstruct piecewise linear x_n of length=33.
Compute: Null vector of a 5×5 Toeplitz matrix; DFTs.
Using: Matrix program listed in longer writeup.
See: www.eecs.umich.edu/~aey/sparse.html
For: Image examples using DFT scaling property.

TINY EXAMPLE [2/2]

Reconstructed x_n and z_n. They match original x_n and z_n: