DEF: $\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$ (compare to $\int x(t)e^{-st}dt$).

ROC: $\{z : X(z)\,\text{converges}\}$; has form $r_1 < |z| < r_2$ for some r_1, r_2.

May be: $0 \le |z| < r$; $0 < |z| < r$; $r < |z| < \infty$; $0 < |z| < \infty$; $0 \le |z| < \infty$

This *matters*; see below. Could usually ignore for Laplace xform.

Finite length signal $x(n) = \{\ldots 0, 0, 3, 1, \underline{4}, 2, 5, 0, 0 \ldots\}(x(0) = 4; \text{finite length} = 5) \rightarrow$

$X(z) = 3z^2 + z + 4 + 2z^{-1} + 5z^{-2} = (3z^4 + z^3 + 4z^2 + 2z + 5)/z^2$.

ROC: $0 < |z| < \infty$ (converges except at 0 and ∞). $\mathcal{Z}\{\delta(n - D)\} = z^{-D}$.

Causal exponentials $\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} = 1/(1 - az^{-1}) = z/(z - a)$.

ROC: $|az^{-1}| < 1 \rightarrow |z| > |a|$ for the series to converge.

EX: $\mathcal{Z}\{(-\frac{1}{3})^n u(n)\} = \frac{1}{1+\frac{1}{3}z^{-1}}$. ROC: $|z| > |-\frac{1}{3}| = \frac{1}{3}$.

Sinusoids $\mathcal{Z}\{\cos(\omega_0 n)u(n)\} = \frac{1}{2}\mathcal{Z}\{e^{j\omega_0 n}u(n)\} + \frac{1}{2}\mathcal{Z}\{e^{-j\omega_0 n}u(n)\} =$ (linear)

$\frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}} = \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$. ROC: $|z| > |e^{j\omega_0}| = 1$.

Anti-causal exponential $\mathcal{Z}\{-a^n u(-n-1)\} = -\sum_{-\infty}^{-1}(\frac{a}{z})^n = -\sum_{n=1}^{\infty}(\frac{z}{a})^n = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$.

ROC: $|a^{-1}z| < 1 \rightarrow |z| < |a|$ for the series to converge.

Anticausal, but has same z-transform as *causal* $a^n u(n)$!

POINT: Need to know ROC to get $x(n)$ from $X(z)$.

Two-sided $\mathcal{Z}\{a^n u(n) + b^n u(-n-1)\} = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}} = \frac{b-a}{a+b-z-abz^{-1}}$.

ROC: $|a| < |z| < |b|$; if $|b| < |a|$ then ROC=empty (*never* converges)!

NOTE: z-xform of sum$\rightarrow$ROC=$\bigcap$(ROCs of each term).

Any $\mathcal{Z}\{x(n)\} = \mathcal{Z}\{x_{causal}(n)\} + \mathcal{Z}\{x_{anticausal}(n)\}\,(x(0) \in x_{causal}(n))$

$= \mathcal{Z}\{\sum A_i p_i^n u(n)\} + \mathcal{Z}\{\sum B_i q_i^n u(-n-1)\}$ for some poles p_i, q_i.

ROC: $|\text{largest pole of } x_{causal}(n)| < |z| < |\text{smallest pole of } x_{anticausal}(n)|$.

Small (magnitude) poles$\leftrightarrow$causal part; large poles$\leftrightarrow$anticausal part.

ROC is always an annulus (ring) whose radii are successive poles.

Also: $\mathcal{Z}\{\frac{1}{n}u(n-1)\} = \sum_{n=1}^{\infty}\frac{z^{-n}}{n} = -\log(1 - z^{-1})$. ROC: $|z| > 1$

if you recognize the infinite series; else no cigar. (But see p. 167.)

Note: Convolution$\leftrightarrow$polynomial multiplication: $\mathcal{Z}\{x(n) * y(n)\} = X(z)Y(z)$.

Eigenfuncs of LTI $z^n \rightarrow \boxed{h(n)} \rightarrow z^n H(z)$: z^n in$\rightarrow$ scaled z^n out.

$h(n) * z^n = \sum h(i)z^{n-i} = z^n \sum h(i)z^{-i} = z^n H(z)$.

z^n plays same role as e^{st} in continuous time.

LE 3.2 PROPERTIES OF THE Z-TRANSFORM

operty	Time Domain	z-Domain	ROC
otation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
nearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
me shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
aling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
me reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
onjugation	$x^*(n)$	$X^*(z^*)$	ROC
eal part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
aginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
fferentiation in the domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
onvolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
orrelation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
itial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
ultiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
rseval's relation	$\sum_{-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		YOU WILL NEVER USE THESE

BLE 3.1 CHARACTERISTIC FAMILIES OF SIGNALS WITH THEIR RRESPONDING ROC

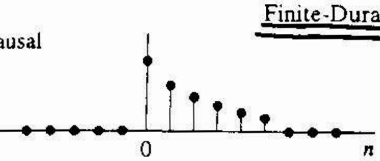
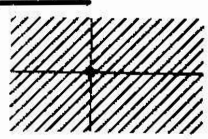
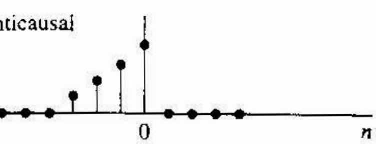
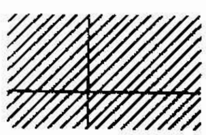
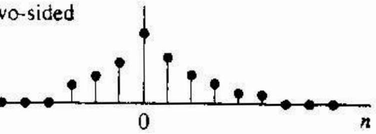
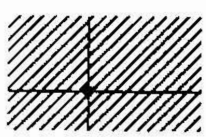
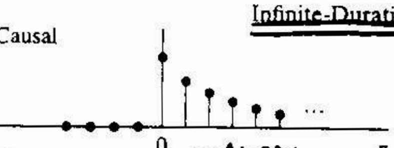
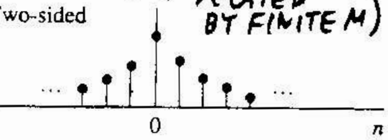
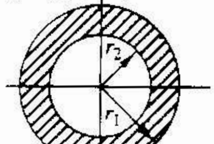
gnal		ROC	
Finite-Duration Signals			
ausal 			Entire z-plane except $z = 0$ PUNCTURED DISK
nticausal 			Entire z-plane except $z = \infty$
wo-sided 			Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals			
Causal 	RIGHT-SIDED (CAUSAL ADVANCED BY FINITE M) : $\infty > \|z\| > r_2$		$\|z\| > r_2$ = LARGEST MAGNITUDE POLE
Anticausal 	LEFT-SIDED (ANTICAUSAL DELAYED BY FINITE M) : $0 < \|z\| < r_1$		$\|z\| < r_1$ = SMALLEST MAGNITUDE POLE
Two-sided 			$r_2 < \|z\| < r_1$

TABLE 3.3 SOME COMMON Z-TRANSFORM PAIRS

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$