

**DEF:**  $\mathcal{Z}^+\{x(n)\} = X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$ , even if  $x(n)$  is noncausal.

Compare to 2-sided z-transform:  $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$ .

**WHY?:** To solve difference equations with nonzero initial conditions.

### PROPERTIES OF 1-SIDED Z-TRANSFORM $X^+(z)$ :

1.  $x(n)$  causal  $\rightarrow$  1-sided z-transform  $X^+(z)$  = 2-sided z-transform  $X(z)$
2. Unique inverse only for causal signals: no info on  $\{x(n), n < 0\}$   
Equal causal parts but unequal anticausal parts  $\rightarrow$  same  $X^+(z)$
3. ROC: Always  $|z| > |p_{max}|$  (largest pole of  $X^+(z)$ ), so don't bother
- 4a.  $D > 0 \rightarrow \mathcal{Z}^+\{x(n-D)\} = z^{-D}(X^+(z) + \sum_{n=1}^D x(-n)z^n)$
- 4b.  $D > 0 \rightarrow \mathcal{Z}^+\{x(n+D)\} = z^{+D}(X^+(z) - \sum_{n=0}^{D-1} x(n)z^{-n})$ 
  - i. Shift right  $\rightarrow$  must add in terms *previously* anticausal
  - ii. Shift left  $\rightarrow$  must subtract off terms *now* anticausal
  - iii. Compare to  $\mathcal{L}\{\frac{d^2x}{dt^2}\} = s^2X(s) - sx(0) - \frac{dx}{dt}(0)$

### SOLVING DIFFERENCE EQNS WITH INITIAL CONDITIONS

**Solve:**  $2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2)$  (Chen p.264)

**Take  $\mathcal{Z}^+$ :**  $2Y^+(z) + 3z^{-1}(Y^+(z) + y(-1)z) + z^{-2}(Y^+(z) + y(-1)z + y(-2)z^2) = (2 + 3z^{-1} + z^{-2})Y^+(z) + [(3y(-1) + y(-2)) + z^{-1}y(-1)] = (1 + z^{-1} - z^{-2})U(z)$

$$\text{Solve: } Y^+(z) = \underbrace{\frac{1 + z^{-1} - z^{-2}}{2 + 3z^{-1} + z^{-2}}U(z)}_{\text{zero-state response}} - \underbrace{\frac{(3y(-1) + y(-2)) + z^{-1}y(-1)}{2 + 3z^{-1} + z^{-2}}}_{\text{zero-input response}}$$

**Plug in:**  $u(n)$  = unit step  $\rightarrow U(z) = U^+(z) = \frac{1}{1-z^{-1}}$ ;  $y(-1) = 2, y(-2) = -1$ :

$$Y^+(z) = \frac{1+z^{-1}-z^{-2}}{(2+3z^{-1}+z^{-2})(1-z^{-1})} - \frac{5+2z^{-1}}{2+3z^{-1}+z^{-2}} = \frac{-4+4z^{-1}+z^{-2}}{2(1+\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})}$$

**Partial fraction**  $Y^+(z) = \frac{4/3}{1+\frac{1}{2}z^{-1}} - \frac{7/2}{1+z^{-1}} + \frac{1/6}{1-z^{-1}}$ . Note no constant term.

$$\mathcal{Z}^{-1}: y(n) = \underbrace{\frac{4}{3}\left(-\frac{1}{2}\right)^n u(n) - \frac{7}{2}(-1)^n u(n)}_{\text{natural response}} + \underbrace{\frac{1}{6}u(n)}_{\text{forced}}$$

**Given:**  $2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2)$  (previous)

$$\text{Soln: } Y(z) = \underbrace{\frac{1 + z^{-1} - z^{-2}}{2 + 3z^{-1} + z^{-2}} U(z)}_{\text{zero-state response(ZSR)}} - \underbrace{\frac{[3y(-1) + y(-2)] + z^{-1}y(-1)}{2 + 3z^{-1} + z^{-2}}}_{\text{zero-input response(ZIR)}}$$

**Poles:** Roots of  $2 + 3z^{-1} + z^{-2} = 0 \rightarrow -1, -\frac{1}{2}$  from denominator of ZSR.

**Modes:** Roots of  $2 + 3z^{-1} + z^{-2} = 0 \rightarrow -1, -\frac{1}{2}$  from denominator of ZIR.

**Huh?**  $\{\text{poles}\} \subseteq \{\text{modes}\}$  since poles can be canceled by zeros of  $H(z)$ .

**Note:** Can also cancel poles by proper choice of input  $u(n)$ .

**EX #1:**  $u(n) = \delta(n) + \delta(n-1) \rightarrow U(z) = 1 + z^{-1} \rightarrow ZSR(z) = \frac{1+z^{-1}-z^{-2}}{2+z^{-1}}$ .

**Note:** Can also cancel modes by proper choice of initial conditions  $y(-1), y(-2)$ .

**EX #2:**  $y(-1) = 1, y(-2) = -2 \rightarrow ZIR(z) = -\frac{1}{2+z^{-1}} \rightarrow zir(n) = -\frac{1}{2}\left(-\frac{1}{2}\right)^n u(n)$ .

**How?** Choose initial conditions  $y(-1), y(-2)$  to cancel the mode at  $-1$ .

**Soln:** Need  $3y(-1) + y(-2) = y(-1)$  so ZIR numerator is  $y(-1)(1 + z^{-1})$ .

**EX #3:** Change one MA-part coefficient so  $\{\text{poles}\} \subset \{\text{modes}\}$ .

**Soln:** Change  $1 + z^{-1} - z^{-2}$  so  $\frac{1+z^{-1}-z^{-2}}{2+3z^{-1}+z^{-2}}$  has pole-zero cancellation.

**EX:** Change ZIR numerator to  $1 + 0z^{-1} - z^{-2} = (1 + z^{-1})(1 - z^{-1})$ .

## CAUSALITY AND BIBO STABILITY

**Fact:** BIBO stable  $\Leftrightarrow \sum |h(n)| < \infty \Leftrightarrow \{|z| = 1\} \subset \text{ROC}$ .

**Fact:** Causal  $\Leftrightarrow \text{ROC} = \{|z| > A\}$  for some constant A. Also Stable if  $|A| < 1$ .

1. Stable AND Causal  $\Leftrightarrow$  Poles inside unit circle  $\Leftrightarrow |p_i| < 1, i = 1 \dots N$ .
2. Stable+Anticausal  $\Leftrightarrow$  Poles outside unit circle  $\Leftrightarrow |p_i| > 1, i = 1 \dots N$ .
3. Stable+2-sided  $\Leftrightarrow \{|z| = 1\} \subset \text{ROC} = \text{annulus}$ .

## INITIAL AND FINAL VALUE THEOREMS AND CORRELATION

**IVT:**  $x(n)$  causal  $\rightarrow \lim_{z \rightarrow \infty} X(z) = x(0)$  since  $X(z) = x(0) + x(1)z^{-1} + \dots$

**FVT:**  $\lim_{n \rightarrow \infty} x(n) = \lim_{z \rightarrow 1} (z-1)X^+(z)$  (1-sided z-xform)(step).

**Auto:**  $r(n) = x(n) * x(-n) = \sum x(i)x(i \pm n) = r(-n)$  (even function).

**Cross:**  $r_{xy}(n) = x(n) * y(-n) = \sum x(i)y(i-n) = r_{yx}(-n) \neq r_{xy}(-n)$ .