WHAT IT: A bandlimited continuous-time signal $x(t)$ can be
DOES: reconstructed perfectly from the *samples* $x(nT)$ of $x(t)$
IF the sampling rate exceeds *twice* the maximum frequency of $x(t)$.
Claude Shannon, U-M alumnus and "father of information theory."

Let signal $x(t)$ have spectrum $X(\omega) = \mathcal{F}\{x(t)\} = 0$ for $|\omega| > B$.
Let $x(t)$ be sampled every $t = nT \leftrightarrow sampling\,rate = S = \frac{2\pi}{T}\,\frac{\text{RADIAN}}{\text{SECOND}}$.

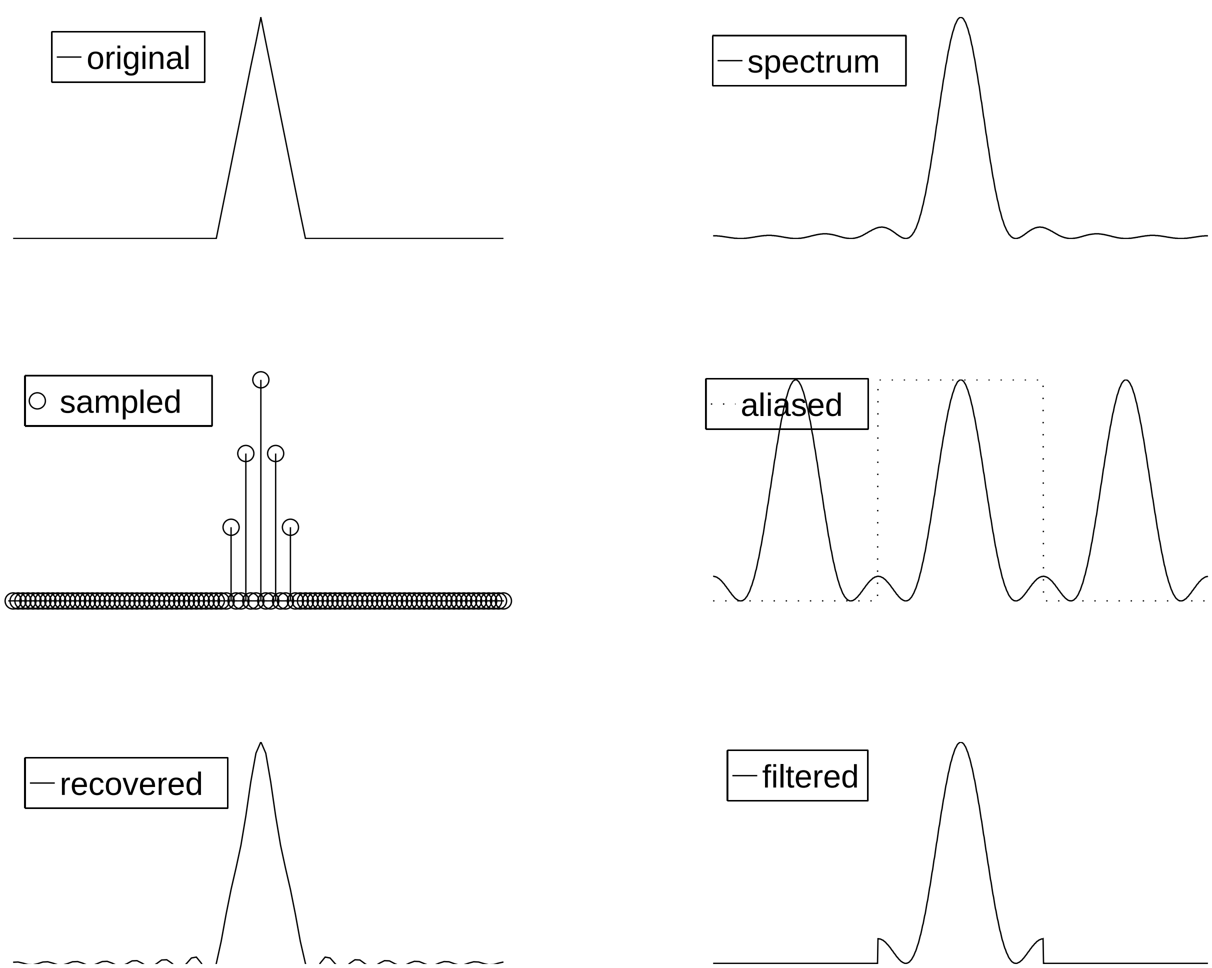
Then can reconstruct $x(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{2B}{S}\frac{\sin B(t-nT)}{B(t-nT)}$
$= [\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)] * \left(\frac{\sin(Bt)}{\pi t}T\right)$ **IF** $S > 2B$=Nyquist rate.

Note that $\mathcal{F}\{\frac{\sin(Bt)}{\pi t}T\} = \begin{cases} T & \text{if } |\omega| < B \\ 0 & \text{if } |\omega| > B \end{cases}$ (low-pass filter=LPF).
So we can implement using: $\{x(nT)\} \times \{\delta(t-nT)\} \rightarrow \boxed{\textbf{LPF}} \rightarrow x(t)$.

WHY THIS IS TRUE: Note $\mathcal{F}\{\sum \delta(t-nT)\} = \sum \frac{2\pi}{T}\delta(\omega - nS)$.
Fourier *series* expansion: $\sum \delta(t-nT) = \sum \frac{1}{T}e^{j2\pi nt/T}$ $(\omega_o = S = \frac{2\pi}{T})$.
Then $\mathcal{F}\{e^{j2\pi nt/T}\} = 2\pi\delta(\omega - nS)$ using modulation property of $\mathcal{F}$.
Periodic AND discrete in time↔Periodic AND discrete in frequency.

$\mathcal{F}\{sampled\,x(t)\} = \mathcal{F}\{\sum x(nT)\delta(t-nT)\} = \mathcal{F}\{x(t)\sum \delta(t-nT)\} =$
$\frac{1}{2\pi}\mathcal{F}\{x(t)\} * \mathcal{F}\{\sum \delta(t-nT)\} = \frac{1}{2\pi}X(\omega) * \sum \frac{2\pi}{T}\delta(\omega - nS) = \frac{1}{T}\sum X(\omega - nS)$

SIGNAL IN: TIME DOMAIN **FREQUENCY DOMAIN**



SAMPLE-AND-HOLD NON-IDEAL SAMPLING

But: An ideal impulse train $\sum \delta(t-nT)$ doesn't exist in the real world!
So: Use *pulse train* $\sum p(t-nT)$ where $p(t)$ is a short pulse: p(t)=rect($\frac{t}{\epsilon}$).
Then: Convolving with $p(t)$ and using $\delta(t-nT) * p(t) = p(t-nT)$ gives
$\mathcal{F}\{\sum x(nT)p(t-nT)\} = \frac{1}{T}P(\omega)\sum X(\omega - nS)$: Just previous$\times P(\omega)$.

So: Reconstruct $x(t)$ from weighted pulse train using $H(\omega)=\frac{rect(\omega/(2B))}{P(\omega)}$.
Also: Using sample-and-hold *interpolation*→Spectrum $P(\omega)\sum X(\omega-nS)$.

COMPUTING SIGNAL BANDWIDTH

EX #1: x(t)=$\frac{3t+7}{t}\sin(2\pi 5t)$: x(t)=$3\sin(2\pi 5t)+7\pi\frac{\sin(2\pi 5t)}{\pi t}$.
EX #2: x(t)=$\sin(2\pi 3t)\sin(2\pi 2t)$: x(t)=$\frac{1}{2}[\cos(2\pi 1t)–\cos(2\pi 5t)]$.
EX #3: x(t)=$\frac{\sin(2\pi 3t)\sin(2\pi 2t)}{t^2}$: $X(\omega) = \frac{1}{2\pi}[rect(\frac{\omega}{6}) * rect(\frac{\omega}{4})]=\frac{1}{2\pi}tri(\frac{\omega}{10})$.
Huh? Convolve length=6 with length=4→length=10.
So: In all three cases, max freq=5 Hz→Nyquist rate=10 Hz→T=$\frac{1}{10}$.

If $S < 2B$? Reconstructed $x(t)$ is **aliased** (see previous page).

EX #1: x(t)=$\sin(2\pi 50t) \rightarrow B = 100\pi$. T=0.01→S=$\frac{2\pi}{0.01}$=$200\pi$=2B=Nyquist.
But: x(nT)=$\sin(2\pi 50(0.01n))$=$\sin(n\pi)$=0! (0 crossings) What's going on?
This: $X(f)=\frac{j}{2}\delta(f+50)–\frac{j}{2}\delta(f-50) \rightarrow \sum X(f-\frac{n}{T}) = 0$! Everything cancels!
So: We need $S > 2B$, not $S \geq 2B$, to ensure reconstruction from samples.

EX #2: x(t)=$\cos(2\pi t)+\cos(10\pi t)$ (1 Hz & 5 Hz)→B=2π(5 Hz)=10π.
Let: T=$\frac{1}{7}$ →S=$\frac{2\pi}{1/7}$=$14\pi < 2B = 20\pi$ (7 Hz sampling rate: undersampled).
Then: $X(\omega) = \pi\delta(\omega-2\pi)+\pi\delta(\omega-10\pi)+\pi\delta(\omega+2\pi)+\pi\delta(\omega+10\pi)$.

Sample: $\frac{1}{T}\sum X(\omega-nS) = 7\sum X(\omega - 14\pi n)$ =periodic extension of this:
$7\pi\delta(\omega-2\pi)+7\pi\delta(\omega-10\pi+14\pi)+7\pi\delta(\omega+2\pi)+7\pi\delta(\omega+10\pi-14\pi)$
$= 7\pi\delta(\omega-2\pi)+7\pi\delta(\omega-4\pi)+7\pi\delta(\omega+2\pi)+7\pi\delta(\omega+4\pi)$.

Get: Cosines at 1 Hz & 2 Hz, NOT 1 Hz & 5 Hz! 2 Hz is *aliased* 5 Hz.
So: *Original* 5 Hz *folded* across 7/2=3.5 Hz becomes an *aliased* 2 Hz.
Folding: $S/2$=**folding** frequency since freqs. above it are folded along it.

Q: How can you tell whether a sampled signal is aliased?
A: Increase the sampling rate as much as possible. Then:

- If the form of the signal doesn't change, it is not aliased.
- If *reduce* sampling rate, at some $\omega = 2B$ the *form* of the signal changes.
- Often a good idea to *oversample* by choosing $S >> 2B$=Nyquist rate.

EX #3: X(ω=$2\pi f$)=0 unless 800< $|f|$ <1200 Hz. Minimum reconstruct rate?
Seems: 2(1200)=2400$\frac{\text{SAMPLE}}{\text{SECOND}} \rightarrow T = \frac{1}{2400}$. **But:** 800$\frac{\text{SAMPLE}}{\text{SECOND}}$ suffices!

Why? Positive freq part has support: [-800,-400],[0,400],[800,1200]...
And: Negative freq part has support: [-1200,-800],[-400,0],[400,800]...
So: Both parts of spectrum are preserved after sampling at 800 Hz!