EECS 451 PRONY'S METHOD: COMPUTING LINE SPECTRA

Given: $x(n) = A_1\cos(\omega_1 n+\theta_1)+...+A_p\cos(\omega_p n+\theta_p)$ for $n=1...3p$

Want: To compute *parameters* $\omega_i, A_i, \theta_i, i=1...p$ from $\{x(1)...x(3p)\}$.

Point: Now are *given* that $x(n)$ is sum of p sinusoids, unlike nonparametric. So problem is only to compute unknown *parameters* ω_i, A_i, θ_i.

DOF: $3p$ nonlinear equations in $3p$ unknowns. Solvable, but looks ugly.

Idea: Rewrite $x(n)=\sum_{i=1}^{p}[B_i e^{j\omega_i n}+B_i^* e^{-j\omega_i n}]$ where $B_i=(A_i/2)e^{j\theta_i}$. This satisfies difference equation of AR order $2p$ with AR coefficients $A(z)=a_0+a_1 z+..a_{2p}z^{2p}=\prod_{i=1}^{p}(z-e^{j\omega_i})(z-e^{-j\omega_i})$.

Props: $z_i^*=\frac{1}{z_i}\rightarrow A(z)=z^{2p}A(1/z)\rightarrow a_i=a_{2p-i}$. Also note $a_0=a_{2p}=1$.

AR: $\sum_{i=0}^{2p}a_i x(n-i)=0$ for $2p+1\le n\le 4p$ ($2p$ initial conditions).

Toeplitz Write as the *Toeplitz* system of equations (constant along diagonals)

system:
$$\begin{bmatrix} x(2p) & x(2p-1) & \dots & x(1)\\ x(2p+1) & x(2p) & \cdots & x(2)\\ \ddots & \ddots & \ddots & \ddots\\ x(4p-1) & x(4p-2) & \cdots & x(2p)\end{bmatrix}\begin{bmatrix}a_1\\a_2\\ \vdots\\ a_{2p}\end{bmatrix}=-\begin{bmatrix}x(2p+1)\\x(2p+2)\\ \vdots\\ x(4p)\end{bmatrix}.$$

But: $a_i=a_{2p-i}, a_0=a_{2p}=1\rightarrow$only need p equations$\rightarrow$need $\{x(1)...x(3p)\}$.

Matrix: Toeplitz+Hankel (constant along diagonals+antidiagonals).

Ex: $x(n)=\cos(\frac{\pi}{2}n)+2\cos(\frac{\pi}{4}n), n=0,1,2,3,4,5$ (can use these as well).

Matrix:
$$\begin{bmatrix} x(3) & x(2) & x(1) & x(0)\\ x(4) & x(3) & x(2) & x(1)\\ x(5) & x(4) & x(3) & x(2)\\ x(6) & x(5) & x(4) & x(3)\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix}=-\begin{bmatrix}x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix}$$
which becomes using $a_i=a_{2p-i}$

Here:
$$\begin{bmatrix}-\sqrt{2}+\sqrt{2} & -1\\ -1-1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}1-3\\ \sqrt{2}-\sqrt{2}\end{bmatrix}\rightarrow\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}-\sqrt{2}\\2\end{bmatrix}.$$

Poly: $[a_0,a_1,a_2,a_3,a_4]=[1,-\sqrt{2},2,-\sqrt{2},1]$ using $a_3=a_1$ and $a_4=a_0=1$.

Modes: Roots of polynomial: $z=e^{\pm j\pi/2},e^{\pm j\pi/4}\rightarrow\omega_1=\pi/2,\omega_2=\pi/4$.

Get A_i: $x(n)=C_1\cos(\frac{\pi}{2}n)+S_1\sin(\frac{\pi}{2}n)+C_2\cos(\frac{\pi}{4}n)+S_2\sin(\frac{\pi}{4}n)$.

Solve:
$$\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -1 & 0 & 0 & 1\\ 0 & -1 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}C_1\\S_1\\C_2\\S_2\end{bmatrix}=\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{bmatrix}=\begin{bmatrix}3\\ \sqrt{2}\\ -1\\ -\sqrt{2}\end{bmatrix};\begin{bmatrix}C_1\\S_1\\C_2\\S_2\end{bmatrix}=\begin{bmatrix}1\\0\\2\\0\end{bmatrix}.$$

Note: Nonlinear problem$\rightarrow$nonlinear problem (roots) we know how to solve.

Given: $y(n) = A\cos(\omega_o n) + w(n)$; A and ω_o are both unknown.
Noise: $w(n)$ is 0-mean *white noise* (if you've taken EECS 401);
$\frac{1}{N}\sum_i w(i)w(i-n) \approx \sigma^2\delta(n)$ (if you haven't taken 401).
Goal: To determine the unknown A and ω_o from the $\{y(n)\}$.
Using: ARMA difference eqns, ZIR, modes, 1-sided z-transform, poles and zeros, autocorrelation, cross-correlation.

Idea: $x(n) = \cos(\omega_o n)u(n)$ is ZIR of AR(2) difference eqn.:
AR(2): $x(n+2) = 2\cos(\omega_o)x(n+1) - x(n)$ with $x(1) = \cos(\omega_o), x(0) = 1$:
1-sided: $[z^2X(z) - x(0)z^2 - x(1)z] = 2\cos(\omega_o)[zX(z) - x(0)] - X(z) \rightarrow$
ZIR: $X(z) = \frac{1-\cos(\omega_o)z^{-1}}{1-2\cos(\omega_o)z^{-1}+z^{-2}} \rightarrow x(n) = \cos(\omega_o n)u(n)$ (text p.174, 200).
Modes: Modes (and poles) at $z = e^{\pm j\omega_o}$ on the unit circle (see also p.353).

ARMA: Substitute $Ax(n) = y(n) - w(n)$ in AR(2) difference eqn.:
$[y(n+2) - w(n+2)] = 2\cos(\omega_o)[y(n+1) - w(n+1)] - [y(n) - w(n)] \rightarrow$
$y(n+2) - 2\cos(\omega_o)y(n+1) + y(n) = w(n+2) - 2\cos(\omega_o)w(n+1) + w(n)$.
ARMA(2,2) difference equation in observations $y(n)$ and noise $w(n)$.

Cross-correlate ARMA(2,2) with $y(n)$ (convolve with $y(-n)$):
$r_{yy}(n+2)$ $-2\cos(\omega_o)r_{yy}(n+1) + r_{yy}(n) \approx r_{wy}(n+2) - 2\cos(\omega_o)r_{wy}(n+1) + r_{wy}(n)$.
Assume: $r_{wy}(n) = r_{wx}(n) + r_{ww}(n) \approx \sigma^2\delta(n)$ (x and w uncorrelated).
Need EECS 401 to justify this, but does happen if N large enough.

Matrix: Arrange above for $n = 0, -1, -2$ as Toeplitz matrix equation

$$\begin{bmatrix} r_{yy}(0) & r_{yy}(1) & r_{yy}(2) \\ r_{yy}(1) & r_{yy}(0) & r_{yy}(1) \\ r_{yy}(2) & r_{yy}(1) & r_{yy}(0) \end{bmatrix} \begin{bmatrix} 1 \\ -2\cos(\omega_o) \\ 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 \\ -2\cos(\omega_o) \\ 1 \end{bmatrix}$$

σ^2 is (unknown) *eigenvalue* of Toeplitz matrix with associated (unknown) *eigenvector* $[1, -2\cos(\omega_o), 1]^T$.
Can show σ^2 is *minimum* eigenvalue of Toeplitz matrix.

Example: We compute $r_{yy}(0) = 3, r_{yy}(1) = 1, r_{yy}(2) = 0$ from the data $\{y(n)\}$:
p.949 $r_{yy}(0) \approx \frac{1}{N}\sum y(i)^2$; $r_{yy}(1) \approx \frac{1}{N}\sum y(i)y(i-1)$; N = #data points.
Matrix: $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ has eigenvalues $3 \pm \sqrt{2}$ and 3 (Matlab: **eig**).
Minimum eigenvalue $3 - \sqrt{2}$ has eigenvector $[1, -\sqrt{2}, 1]^T$.
$-2\cos(\omega_o) = -\sqrt{2} \rightarrow \omega = \pm\pi/4 \rightarrow$ poles at $e^{\pm j\pi/4}$.
Soln: Frequency $\omega_o = \pi/4$; $\sigma^2 = 3 - \sqrt{2}$.