Linear $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\text{ARG}[X(e^{j\omega})]}$ and $\text{ARG}[X(e^{j\omega})] = -\omega D$.

Phase: Delay: $\text{DTFT}\{\delta(n-D)\} = e^{-j\omega D} \rightarrow \text{ARG} = -\omega D \rightarrow$ linear phase.

$D$ not an integer $\rightarrow x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega D} e^{j\omega n} d\omega = \frac{\sin\pi(n-D)}{\pi(n-D)}$.

Group $	au_g(\omega) = \text{grd}[X(e^{j\omega})] = -\frac{d}{d\omega} \text{ARG}[X(e^{j\omega})]$ = slope of phase.

delay: Phase must be unwrapped so that it is a continuous function.

Narrow $x(n) =$ narrowband $: X(e^{j\omega}) = 0$ unless $|\omega| < \Delta$.

band: $w(n) = x(n) \cos(\omega_o n) \rightarrow W(e^{j\omega}) = 0$ unless $\omega_o - \Delta < |\omega| < \omega_o + \Delta$.

Then: $[w(n) \rightarrow \overline{H_{ap}(z)} \rightarrow y(n)] \rightarrow y(n) \approx x(n - \tau_g) \cos(\omega_o n - \omega_o \tau_g - \theta_o)$

where: $\text{ARG}[H_{ap}(e^{j\omega})] \approx -\omega \tau_g - \theta_o$ for $\omega_o - \Delta < |\omega| < \omega_o + \Delta$.

Delay: $	au_g(\omega_0)$ is ”time delay” for signal components at $\omega \approx \omega_o$.

Inverse $x(n) \rightarrow \overline{[h(n)]} \rightarrow \overline{[y(n)]} \rightarrow \overline{[i(n)]} \rightarrow x(n)$: $i(n)$ ”undoes” $h(n)$.

system: $h(n) \ast i(n) = \delta(n) \rightarrow I(z) = 1/H(z)$. $H(z) = \frac{N(z)}{D(z)} \rightarrow I(z) = \frac{D(z)}{N(z)}$.

Stable? Stable $i(n)$ exists iff $H(z)$ has no zeros on the unit circle.

Choose ROC of $I(z) = \frac{D(z)}{N(z)}$ so that it includes the unit circle.

Causal? Stable-causal $h(n)$ has stable-causal $i(n)$ iff $H(z)$ has all poles+zeros inside unit circle $\Leftrightarrow H(z)$ minimum phase.

Min.: $x(n)$ minimum phase $\Leftrightarrow$ all poles+zeros inside unit circle $\Leftrightarrow$ Phase $x(n)$ stable and causal and has a stable and causal inverse.

Factorization Any $X(z)$ with no poles or zeros on the unit circle factors:

$X(z) = \overline{X_{min}(z)}\overline{X_{max}(z)} = \overline{X_{min2}(z)}\overline{X_{allpass}(z)}$.

#1: Separate poles+zeros inside/outside unit circle to get $X_{min}(z), X_{max}(z)$.

#2: ”Flip” poles+zeros outside unit circle to inside to get $X_{min2}(z)$.

Magnitude $H(z)$ (represents comm. channel) has zeros outside unit circle.

compensation: No stable-causal inverse filter: $H(z)$ not minimum phase. BUT:

$1/H_{min2}(z)$ is stable-causal and compensates magnitude (but not phase).

Properties: $\text{ARG}[X_{ap}(e^{j\omega})] < 0$ and $\text{grd}[X_{ap}(e^{j\omega})] > 0$ for $0 < \omega < \pi$ (see p.352).

Phase lag: $\text{ARG}[X(e^{j\omega})] = \text{ARG}[X_{min2}(e^{j\omega})] + \text{ARG}[X_{ap}(e^{j\omega})] < \text{ARG}[X_{min2}(e^{j\omega})]$.

Group lag: $\text{grd}[X(e^{j\omega})] = \text{grd}[X_{min2}(e^{j\omega})] + \text{grd}[X_{ap}(e^{j\omega})] > \text{grd}[X_{min2}(e^{j\omega})]$. 
Given: \( x(n) \rightarrow |H(z)| \rightarrow y(n) \) where \( y(n) = x(n) - 2x(n-1) \leftrightarrow H(z) = \frac{z^{-2}}{z} \).

This could be a communications channel with a large echo.

Goal: Determine a stable and causal inverse filter \( i(n) \) for \( h(n) \).

But: \( I(z) = \frac{1}{H(z)} = \frac{z}{z-2} \rightarrow i(n) = 2^n u(n) \) or \( i(n) = -2^n u(-n - 1) \).

There is no stable AND causal inverse filter! What do we do?

Idea: We can’t undo \( H(z) \). We CAN undo \(|H(e^{j\omega})|\), but not \( \text{ARG}[H(e^{j\omega})] \).

How? Write \( H(z) = H_{\text{min}}(z) H_{\text{ap}}(z) = \frac{z^{-2}}{z} = \left( \frac{2z^{-2}}{z} \right) \left( \frac{1}{2} \right) \).

Huh? Reflect zero outside the unit circle to inside the unit circle.

Then: \( |H(e^{j\omega})| = |H_{\text{min}}(e^{j\omega})| \cdot |H_{\text{ap}}(e^{j\omega})| = |H_{\text{min}}(e^{j\omega})| \) (see overleaf).

So? \( H_{\text{min}}(z) \), unlike \( H(z) \), HAS a stable and causal inverse filter.

Use: \( i(n) = Z^{-1} \left\{ \frac{1}{H_{\text{min}}(z)} \right\} = Z^{-1} \left\{ \frac{1}{2} \frac{z}{z-2} \right\} = (\frac{1}{2})^{n+1} u(n) \) stable & causal.

Then: \( x(n) \rightarrow \left| \frac{z^{-2}}{z} \right| \rightarrow y(n) \rightarrow \left| \frac{1}{2} \frac{z}{z-2} \right| \rightarrow w(n) \) approximation to \( x(n) \).

where: \( |W(e^{j\omega})| = |X(e^{j\omega})| \) but \( \text{ARG}[W(e^{j\omega})] \neq \text{ARG}[X(e^{j\omega})] \).

This is the best we can do, under constraints of stability and causality.

**PRINCIPLE OF THE ARGUMENT**

Fact: Let \( H(z) \) have \( Z \) zeros and \( P \) poles INSIDE the unit circle.

Let: \( H(z)|_{z=e^{j\omega}} = H(e^{j\omega}) = |H(e^{j\omega})|e^{j\text{ARG}[H(e^{j\omega})]} \) as usual.

Let: \( \text{ARG}[H(e^{j\omega})] \) be the unwrapped phase response of the system.

Then: \( \text{ARG}[H(e^{j2\pi})] = 2\pi(Z-P) \). If know \( P \) (usually do), can find \( Z \).

1. Unwrapped phase at \( \omega = 2\pi \rightarrow \# \text{zeros inside unit circle} \).
2. Don’t even need the magnitude response \(|H(e^{j\omega})|\)! Phase is enough!
3. See the back side of Problem Set #6 for an example of this.
4. If the unwrapped phase of a stable system at \( \omega = 2\pi \) is zero, and the system is causal but \( h(0) \neq 0 \), then it is also minimum phase, since \#poles=\#zeros and all poles are inside the unit circle.
Energy lag: $E[x](n) < E[x_{min2}](n)$ where $E[x](n) = \sum_{i=0}^{n} |x(i)|^2$. See below.

Proof: Consider effect of flipping a zero $z_o$ from inside to outside unit circle:

Let: $X_{min2}(z) = Q(z)(1-z_o z^{-1})$ and $X(z) = Q(z)(z^{-1} - z_o^*)$ where $|z_o| < 1$.

Note: $|X_{min2}(e^{j\omega})| = |X(e^{j\omega})|$ and $Q(z)$ minimum phase $\Rightarrow q(n)$ causal.

Then: $|x_{min2}^2(n)| = |q(n) - z_o q(n-1)|^2$ and $|x^2(n)| = |q(n-1) - z_o^* q(n)|^2$.

And: $\sum_{i=0}^{n} |x_{min2}^2(i)| = \sum_{i=0}^{n-1} [q^2(i)(1+z_o^2) - 2Re[z_o q(i) q^*(i+1)] + |q^2(n)|$.

And: $\sum_{i=0}^{n} |x^2(i)| = \sum_{i=0}^{n-1} [q^2(i)(1+z_o^2) - 2Re[z_o q(i) q^*(i+1)] + |z_o^2||q^2(n)|$.

So: $\sum_{i=0}^{n} |x_{min2}^2(i)| > \sum_{i=0}^{n} |x^2(i)|$ since $|z_o^2| < 1$. Q.E.D.
clear
X1=[1.56, 2.89, 3.39, 2.19, 0.81];
X2=[0.81, 2.19, 3.39, 2.89, 1.56];
X3=[1.26, 2.51, 3.50, 2.58, 1.00];
X4=[1.00, 2.58, 3.50, 2.51, 1.26];
A1=unwrap(angle(fft(X1,256)));
A2=unwrap(angle(fft(X2,256)));
A3=unwrap(angle(fft(X3,256)));
A4=unwrap(angle(fft(X4,256)));
T=1:256;
plot(T,A1,'-',T,A2,'.',T,A3,'--',T,A4,'-.'