EECS 451 GROUP DELAY, INVERSE, MINIMUM PHASE

Linear phase: $X(e^{j\omega}) = |X(e^{j\omega})|e^{jARG[X(e^{j\omega})]}$ and $ARG[X(e^{j\omega})] = -\omega D$.
Delay: $\text{DTFT}\{\delta(n-D)\} = e^{-j\omega D} \rightarrow ARG = -\omega D \rightarrow$linear phase.
D not an integer$\rightarrow x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega D}e^{j\omega n}d\omega = \frac{\sin\pi(n-D)}{\pi(n-D)}$.

Group delay: $\tau_g(\omega) = grd[X(e^{j\omega})] = -\frac{d}{d\omega}ARG[X(e^{j\omega})]$=–slope of phase.
Phase must be *unwrapped* so that it is a continuous function.

Narrow band: $x(n) = narrowband : X(e^{j\omega}) = 0$ unless $|\omega| < \Delta$.
$w(n) = x(n)\cos(\omega_o n) \rightarrow W(e^{j\omega}) = 0$ unless $\omega_o - \Delta < |\omega| < \omega_o + \Delta$.

Then: $[w(n) \rightarrow \boxed{H_{ap}(z)} \rightarrow y(n)] \rightarrow y(n) \approx x(n-\tau_g)\cos(\omega_o n - \omega_o\tau_g - \theta_o)$
where: $ARG[H_{ap}(e^{j\omega})] \approx -\omega\tau_g - \theta_o$ for $\omega_o - \Delta < |\omega| < \omega_o + \Delta$.
Delay: $\tau_g(\omega_0)$ is "time delay" for signal components at $\omega \approx \omega_o$.

Inverse system: $x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) \rightarrow \boxed{i(n)} \rightarrow x(n)$: $i(n)$ "undoes" $h(n)$.
$h(n) * i(n) = \delta(n) \rightarrow I(z) = 1/H(z)$. $H(z) = \frac{N(z)}{D(z)} \rightarrow I(z) = \frac{D(z)}{N(z)}$.

Stable? Stable $i(n)$ exists iff $H(z)$ has no zeros on the unit circle.
Choose ROC of $I(z) = \frac{D(z)}{N(z)}$ so that it includes the unit circle.

Causal? Stable+causal $h(n)$ has stable+causal $i(n)$ iff
$H(z)$ has all poles+zeros inside unit circle$\Leftrightarrow$ $H(z)$ *minimum phase.*

Min.: Phase $x(n)$ minimum phase$\Leftrightarrow$all poles+zeros inside unit circle$\leftrightarrow$
$x(n)$ stable and causal and has a stable and causal inverse.

Factorization Any $X(z)$ with no poles or zeros *on* the unit circle factors:
$X(z) = \underbrace{X_{min}(z)}_{\substack{min\\phase}}\underbrace{X_{max}(z)}_{\substack{max\\phase}} = \underbrace{X_{min2}(z)}_{\substack{min\\phase}}\underbrace{X_{ap}(z)}_{\substack{all\\pass}}$.

#1: Separate poles+zeros inside/outside unit circle to get $X_{min}(z), X_{max}(z)$.
#2: "Flip" poles+zeros outside unit circle to inside to get $X_{min2}(z)$.

Magnitude compensation: $H(z)$ (represents comm. channel) has zeros outside unit circle.
No stable+causal inverse filter: $H(z)$ not minimum phase. BUT:
$1/H_{min2}(z)$ is stable+causal and compensates magnitude (but not phase).

Properties: $ARG[X_{ap}(e^{j\omega})] < 0$ and $grd[X_{ap}(e^{j\omega})] > 0$ for $0 < \omega < \pi$ (see p.352).

Phase lag: $ARG[X(e^{j\omega})] = ARG[X_{min2}(e^{j\omega})] + ARG[X_{ap}(e^{j\omega})] < ARG[X_{min2}(e^{j\omega})]$.
Group lag: $grd[X(e^{j\omega})] = grd[X_{min2}(e^{j\omega})] + grd[X_{ap}(e^{j\omega})] > grd[X_{min2}(e^{j\omega})]$.

Given: $x(n) \to \boxed{H(z)} \to y(n)$ where $y(n) = x(n) - 2x(n-1) \leftrightarrow H(z) = \frac{z-2}{z}$.
This could be a communications channel with a large echo.

Goal: Determine a stable and causal inverse filter $i(n)$ for $h(n)$.

But: $I(z) = \frac{1}{H(z)} = \frac{z}{z-2} \to i(n) = 2^n u(n)$ or $i(n) = -2^n u(-n-1)$.
There is no stable AND causal inverse filter! What do we do?

Idea: We can't undo $H(z)$. We CAN undo $|H(e^{j\omega})|$, but not $ARG[H(e^{j\omega})]$.

How? Write $H(z) = H_{min}(z)H_{ap}(z) = \frac{z-2}{z} = \left(2\frac{z-\frac{1}{2}}{z}\right)\left(\frac{1}{2}\frac{z-2}{z-\frac{1}{2}}\right)$.

Huh? Reflect zero outside the unit circle to inside the unit circle.
Zero at 2 becomes a zero at $\frac{1}{2}$. Note the all-pass filter.

Then: $|H(e^{j\omega})| = |H_{min}(e^{j\omega})| \cdot |H_{ap}(e^{j\omega})| = |H_{min}(e^{j\omega})|$ (see overleaf).

So? $H_{min}(z)$, unlike $H(z)$, HAS a stable and causal inverse filter.

Use: $i(n) = \mathcal{Z}^{-1}\{\frac{1}{H_{min}(z)}\} = \mathcal{Z}^{-1}\{\frac{1}{2}\frac{z}{z-\frac{1}{2}}\} = (\frac{1}{2})^{n+1}u(n)$ stable & causal.

Then: $x(n) \to \boxed{\frac{z-2}{z}} \to y(n) \to \boxed{\frac{1}{2}\frac{z}{z-\frac{1}{2}}} \to w(n)$ approximation to $x(n)$.

where: $|W(e^{j\omega})| = |X(e^{j\omega})|$ but $ARG[W(e^{j\omega})] \neq ARG[X(e^{j\omega})]$.
This is the best we can do, under constraints of stability and causality.

PRINCIPLE OF THE ARGUMENT

Fact: Let $H(z)$ have Z zeros and P poles INSIDE the unit circle.

Let: $H(z)|_{z=e^{j\omega}} = H(e^{j\omega}) = |H(e^{j\omega})|e^{jARG[H(e^{j\omega})]}$ as usual.

Let: $ARG[H(e^{j\omega})]$ be the *unwrapped* phase response of the system.

Then: $ARG[H(e^{j2\pi})] = 2\pi(Z - P)$. If know P (usually do), can find Z.

---

1. Unwrapped phase at $\omega = 2\pi \to$ #zeros inside unit circle.
2. Don't even need the magnitude response $|H(e^{j\omega})|$! Phase is enough!
3. See the back side of Problem Set #6 for an example of this.
4. If the *unwrapped phase* of a stable system at $\omega = 2\pi$ is zero, and the system is causal but $h(0) \neq 0$, then it is also minimum phase, since #poles=#zeros and all poles are inside the unit circle.

nergy lag: $E[x](n) < E[x_{min2}](n)$ where $E[x](n) = \sum_{i=0}^{n} |x(i)|^2$. See below.

Proof: Consider effect of flipping a zero z_o from inside to outside unit circle:

Let: $X_{min2}(z) = Q(z)(1 - z_o z^{-1})$ and $X(z) = Q(z)(z^{-1} - z_o^*)$ where $|z_o| < 1$.

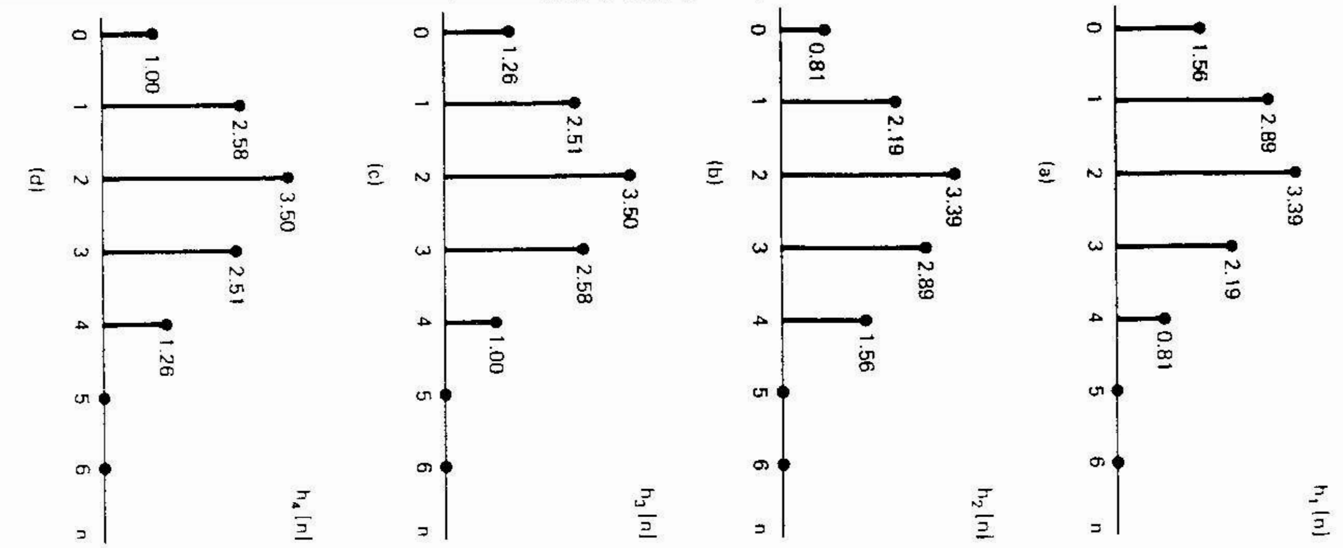
Note: $|X_{min2}(e^{j\omega})| = |X(e^{j\omega})|$ and $Q(z)$ minimum phase$\rightarrow q(n)$ causal.

Then: $|x_{min2}^2(n)| = |q(n) - z_o q(n-1)|^2$ and $|x^2(n)| = |q(n-1) - z_o^* q(n)|^2$.

And: $\sum_{i=0}^{n} |x_{min2}^2(i)| = \sum_{i=0}^{n-1} [q^2(i)(1 + |z_o|^2) - 2Re[z_o q(i) q^*(i+1)]] + |q^2(n)|$.

And: $\sum_{i=0}^{n} |x^2(i)| = \sum_{i=0}^{n-1} [q^2(i)(1 + |z_o|^2) - 2Re[z_o q(i) q^*(i+1)]] + |z_o^2||q^2(n)|$.

So: $\sum_{i=0}^{n} |x_{min2}^2(i)| > \sum_{i=0}^{n} |x^2(i)|$ since $|z_o^2| < 1$. Q.E.D.



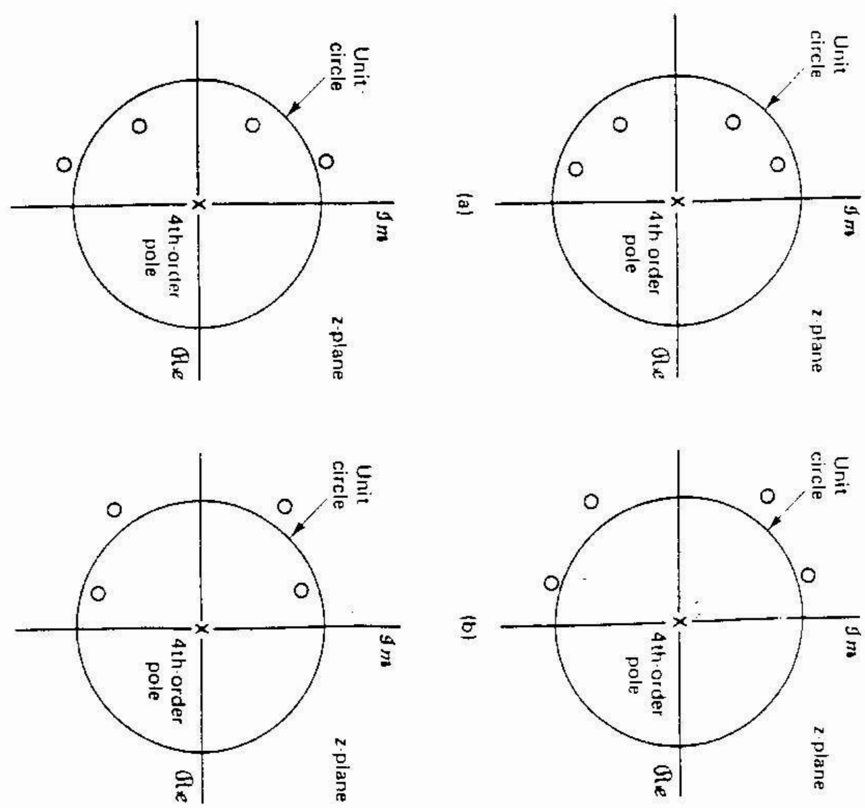


Figure 5.27 Four systems, all having the same frequency-response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.

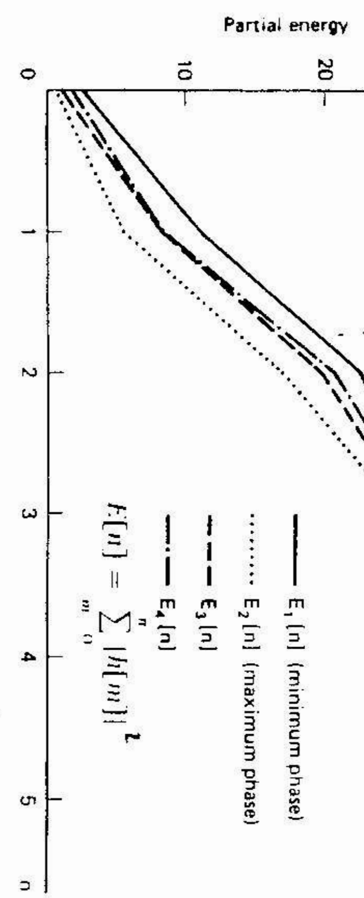


Figure 5.29 Partial energies for the four sequences of Fig. 5.27. (Note that $E_1[n]$ is for the minimum-phase sequence $h_1[n]$, and $E_2[n]$ is for the maximum-phase

```
clear
X1=[1.56,2.89,3.39,2.19,0.81];
X2=[0.81,2.19,3.39,2.89,1.56];
X3=[1.26,2.51,3.50,2.58,1.00];
X4=[1.00,2.58,3.50,2.51,1.26];
A1=unwrap(angle(fft(X1,256)));
A2=unwrap(angle(fft(X2,256)));
A3=unwrap(angle(fft(X3,256)));
A4=unwrap(angle(fft(X4,256)));
T=1:256;
plot(T,A1,'-',T,A2,'.',T,A3,'--',T,A4,'-.');
```

WHY "MINMMUM PHASE"? ACTUALLY, SHOULD BE "MINIMUM PHASE LAG." THE UNWRAPPED PHASES FOR THE 4 SIGNALS ON THE REVERSE SIDE OF "GROUP DELAY..."