1. If x(n) real: ($\exists$ zero at $z_o \rightarrow \exists$ zero at z_o^*); ($\exists$ pole at $p_o \rightarrow \exists$ pole at p_o^*).
Why? Zeros of real polynomials always occur in complex conjugate pairs.

2. If x(n) linear phase: ($\exists$ zero $z_o \rightarrow \exists$ zero $\frac{1}{z_o}$); ($\exists$ pole $p_o \rightarrow \exists$ pole $\frac{1}{p_o}$).
Why? $[x(n) = x(-n)] \leftrightarrow [X(z) = X(\frac{1}{z})] \leftrightarrow [(X(z_o) = 0) \leftrightarrow (X(\frac{1}{z_o}) = 0)]$.

or: $[x(n) = -x(-n)] \leftrightarrow [X(z) = -X(\frac{1}{z})] \leftrightarrow [X(z) = (z - \frac{1}{z})Y(z), Y(z) = Y(\frac{1}{z})]$
since ± 1 are clearly zeros of $X(z) = \ldots cz^2 + bz + 0 - \frac{b}{z} - \frac{c}{z^2} - \ldots$
Then: $[x(n)$ **linear phase**$] \leftrightarrow [x(n) = y(n-D)$ and $y(n) = \pm y(-n)]$.

So? $h(n)$ has **linear phase** $\leftrightarrow h(n) = g(n-D), g(n) = \pm g(-n)$.
and: $g(n)$ has **constant phase** (0 or π) so no signal distortion.
Hence: We want filters to have linear phase (at least in passband), so they won't distort the signal phase, only its amplitude.

3. If x(n) all-pass: ($\exists$ zero $z_o \rightarrow \exists$ pole $\frac{1}{z_o^*}$); ($\exists$ pole $p_o \rightarrow \exists$ zero $\frac{1}{p_o^*}$).
Why? $X(z) = \frac{1}{z_o^*}\frac{z-z_o}{z-\frac{1}{z_o^*}} \rightarrow X(z)X^*(\frac{1}{z^*}) = \frac{1+|z_o^2|-zz_o^*-\frac{z_o}{z}}{1+|z_o^2|-zz_o^*-\frac{z_o}{z}} = 1$ (try it!).
Then: $z = e^{j\omega} \rightarrow \frac{1}{z^*} = e^{j\omega} \rightarrow X(e^{j\omega})X^*(e^{j\omega}) = |X(e^{j\omega})|^2 = 1$.
Note: If $x(n)$ is also real, then zeros are in conjugate pairs; also poles.

EX #1: $h(n) = \{1, \underline{0}, 1\} \rightarrow H(e^{j\omega}) = e^{j\omega} + e^{-j\omega} = 2\cos\omega$.
Then: $|H(e^{j\omega})| = 2|\cos\omega|$ and $ARG[H(e^{j\omega})] = 0, |\omega| < \frac{\pi}{2}; \pi, \frac{\pi}{2} < |\omega| < \pi$.

EX #2: $h(n) = \{\underline{1}, 0, 1\} \rightarrow H(e^{j\omega}) = 1 + e^{-j2\omega} = 2e^{-j\omega}\cos\omega$.
Then: $|H(e^{j\omega})| = 2|\cos\omega|$ and $ARG[H(e^{j\omega})] = -\omega, |\omega| < \frac{\pi}{2}$: **linear phase**.

EX #3: $h(n) = \{1, \underline{0}, -1\} \rightarrow H(e^{j\omega}) = e^{j\omega} - e^{-j\omega} = 2j\sin\omega$.
Then: $|H(e^{j\omega})| = 2|\sin\omega|$ and $ARG[H(e^{j\omega})] = \frac{\pi}{2}SGN[\omega], |\omega| < \pi$

EX #4: $h(n) = \{\underline{1}, 0, -1\} \rightarrow H(e^{j\omega}) = 1 - e^{-j2\omega} = 2je^{-j\omega}\sin\omega$.
Then: $|H(e^{j\omega})| = 2|\sin\omega|$ and $ARG[H(e^{j\omega})] = (\frac{\pi}{2} - \omega), 0 < \omega < \pi$.

1. Assumes slope of linear phase is integer; else noninteger time delay!
2. Time delay or linear phase does not affect magnitude response.
3. $ARG[2je^{-j\omega}\sin\omega] = ARG[2] + ARG[j] + ARG[e^{-j\omega}] + ARG[\sin\omega] =$ $0 + \frac{\pi}{2} - \omega + 0$ if $0 < \omega < \pi$; $\quad 0 + \frac{\pi}{2} - \omega - \pi$ if $-\pi < \omega < 0$.

Given: The *autocorrelation* $r(n) = x(n) * x(-n)$ of an unknown signal $x(n)$.
Goal: All solutions $x(n)$ whose autocorrelation $x(n)*x(-n)$ is the given $r(n)$.
Same as: Given $|X(e^{j\omega})|^2$, compute $X(e^{j\omega})$ ("phase retrieval from magnitude").
Why? Phase is often distorted in practice; blind deconvolution of even $h(n)$.
Also: Need this to understand DSP concept of "*minimum phase*" signals.

Properties of r(n): r(n)=r(-n)$\rightarrow R(z) = R(1/z) \rightarrow$zeros in *complex conjugate quadruples:* z_0 is a zero of $R(z) \rightarrow z_0^*, 1/z_o, 1/z_o^*$ are then also zeros of $R(z)$.

Positive definite: $R(e^{j\omega})$ is always real since $r(n)$ even; $R(e^{j\omega}) > 0 \Leftrightarrow$positive definite. If **not** positive definite, some zeros lie *on* unit circle$\rightarrow z_0^* = 1/z_o$.

Assume: **1.** The given $r(n)$ is *positive definite*; $r(n)$ has length $4M+1$;
2. All zeros of $R(z)$ are complex (none of them lies on the real axis).

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1. Compute $4M$ zeros of $R(z)$; will be in M groups $\{z_i, z_i^*, 1/z_i, 1/z_i^*\}$.
2. For each $i = 1 \ldots M$, choose **either** pair $\{z_i, z_i^*\}$ **or** pair $\{1/z_i, 1/z_i^*\}$: $x(n)$ real$\rightarrow$we must choose zeros in complex conjugate pairs.
3. Compute $X_k(z) = C_k \prod_{i=1}^{M}(z - [z_i \,\mathbf{or}\, 1/z_i])(z - [z_i^* \,\mathbf{or}\, 1/z_i^*])$, where C_k is a *scale factor* chosen so that $X_k(z)X_k(1/z) = R(z)$.
4. Properties of the 2^M solutions $X_k(z), k = 1 \ldots 2^M$ computed above:
   a. Each solution is product of M quadratic terms$\rightarrow$length=$2M+1$;
   b. 1 choice for each of M complex conj. quadruples$\rightarrow 2^M$ solutions;
   c. Half of the solutions are *time reversals* $x(-n)$ of the other half;
   d. Could multiply each by -1 to get another solution; don't count.

Example: $r(n) = \{4,0,0,0,17,0,0,0,4\} \rightarrow R(z) = 4z^4 + 17 + 4z^{-4} \rightarrow M = 2$.
$R(e^{j\omega}) = 17 + 8\cos(4\omega) > 0$ for all $\omega \rightarrow r(n)$ positive definite.
Zeros: $R(z)$ has zeros at: $\{\frac{1}{\sqrt{2}}e^{\pm j\pi/4}, \frac{1}{\sqrt{2}}e^{\pm j3\pi/4}, \sqrt{2}e^{\pm j\pi/4}, \sqrt{2}e^{\pm j3\pi/4}\}$.

$$X_1(z) = 4(z - \tfrac{1}{\sqrt{2}}e^{j\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{-j\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{j3\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{-j3\pi/4}) = 4z^4 + 1.$$
$x_1(n) = \{4,0,0,0,1\}$. Scale factor 4 needed so that $x_1(n) * x_1(-n) = r(n)$.
$$X_2(z) = 2(z - \tfrac{1}{\sqrt{2}}e^{j\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{-j\pi/4})(z - \sqrt{2}e^{j3\pi/4})(z - \sqrt{2}e^{-j3\pi/4}).$$
$x_2(n) = \{2,2,1,-2,2\}$. Scale factor 2 needed so that $x_2(n) * x_2(-n) = r(n)$.
$$X_3(z) = 2(z - \sqrt{2}e^{j\pi/4})(z - \sqrt{2}e^{-j\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{j3\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{-j3\pi/4}).$$
$x_3(n) = \{2,-2,1,2,2\}$. Note $x_3(n) = x_2(-n)$ (time reversal).
$$X_4(z) = (z - \sqrt{2}e^{j\pi/4})(z - \sqrt{2}e^{-j\pi/4})(z - \sqrt{2}e^{j3\pi/4})(z - \sqrt{2}e^{-j3\pi/4}) = z^4 + 4.$$
$x_4(n) = \{1,0,0,0,4\}$. Note $x_4(n) = x_1(-n)$ (time reversal).