Goal: Design IIR (Infinite Impulse Response) digital filters
From: Butterworth, Chebyschev and elliptic analog filters.
Why? #coefficients$\rightarrow \infty$, so too hard to do directly.

Given: Analog lowpass (or whatever) filter $H_a(s)$.
Want: Digital lowpass (or whatever) filter $H(z)$.
Mapping: Clearly need to substitute *mapping* $s \rightarrow z$
so that: (1) $s = j\Omega \rightarrow z = e^{j\omega}$; (2) $Re[s] < 0 \rightarrow |z| < 1$ (stability preserved).

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1. **Backward difference:** $s \Leftrightarrow \frac{d}{dt} \approx \frac{x(nT)-x((n-1)T)}{T}$. Need $T << 1$.

Mapping: $s = (1 - z^{-1})/T \rightarrow H(z) = H_a(s = (1 - z^{-1})/T)$. Example: p. 670.
Props: (1) Imag axis NOT mapped to unit circle; (2) stability preserved.
Imag axis$\rightarrow$circle $|z - \frac{1}{2}| = \frac{1}{2}$; tangent to $|z| = 1$ for $z \approx 1$ (DC).

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2. **Impulse invariance:** $h(n) = h_a(t = nT)$. *Sample impulse response.*

Cont: $H_a(s) = \sum_{k=1}^{p} \frac{c_k}{s-p_k} \Leftrightarrow h_a(t) = \sum_{k=1}^{p} c_k e^{p_k t} u(t)$. Substitute $t = nT$:
Disc: $H(z) = \sum_{k=1}^{p} \frac{c_k}{1-e^{p_k T} z^{-1}} \Leftrightarrow h(n) = \sum_{k=1}^{p} c_k e^{p_k T n} u(n) = h_a(t = nT)$.

Sampling: Can also interpret as *sampling* $h(t)$ (see p. 671-673; not worth it).
"Mapping": $z = e^{sT}$ maps poles, *not zeros*! Apply to partial fraction of $H_a(s)$.

Example: $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9} = \frac{1/2}{s-(-0.1+j3)} + \frac{1/2}{s-(-0.1-j3)} \Leftrightarrow h_a(t) = e^{-0.1t}\cos(3t)u(t)$.
(p. 675) $H(z) = \frac{1/2}{1-e^{(-0.1+j3)T}z^{-1}} + \frac{1/2}{1-e^{(-0.1-j3)T}z^{-1}} \Leftrightarrow h(n) = e^{-0.1nT}\cos(3nT)u(n)$.

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3. **Bilinear Transform:** $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$. *"Bilinear"=ratio of linears.*

On the imaginary axis: $s = j\Omega$ and $z = e^{j\omega} \rightarrow \Omega = \frac{2}{T}\tan\frac{\omega}{2}$.
Why? Can be derived from trapezoidal rule for integration (previous HO).
Props: (1) Imag axis mapped to unit circle; (2) stability preserved (p. 678).

Example: $H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$. Want resonant freq. $\Omega = 4$ to map to $\omega = \frac{\pi}{2}$.
(p. 679) $\Omega = \frac{2}{T}\tan\frac{\omega}{2} \rightarrow 4 = \frac{2}{T}\tan\frac{\pi}{4} \rightarrow T = \frac{1}{2}$.
Insert $s = 4\frac{1-z^{-1}}{1+z^{-1}}$ in $H_a(s) \rightarrow H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.0006z^{-1}+0.975z^{-2}}$. Poles: $0.987e^{\pm j\pi/2}$.

Frequency Warping: Need to *prewarp* Ω due to $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$:
Example: Design one-pole lowpass filter with 3 dB bandwidth of 0.2π.
(p. 680) $\Omega = \frac{2}{T}\tan(0.1\pi) = \frac{0.65}{T} \rightarrow H_a(s) = \frac{0.65/T}{s+0.65/T}$ (one-pole analog LPF).
Insert $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \rightarrow H(z) = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$. T cancels! $|H(e^{j0.2\pi})| = 0.707$.

Given: $H_a(s) = 1000/(s+1000) \Leftrightarrow h_a(t) = 1000e^{-1000t}u(t)$ (1-pole filter).
Goal: Map this analog filter to 3 digital IIR filters using the following: Backward differences, impulse invariance, bilinear transformation.

Analog: $|H(j\Omega)| = 1000/\sqrt{\Omega^2 + 10^6}$. 3 dB freq=1000 radians/sec≈160 Hz.
Sample: Suggests sampling rate of 1000 Hz→ $T = 0.001$ seconds=1 ms.

Backward difference $s = \frac{1-z^{-1}}{T} = 1000(1-z^{-1}) \rightarrow H(z) = \frac{1000}{1000(1-z^{-1})+1000} = 1/(2-z^{-1})$.
$H(e^{j0}) = 1/(2-1) = 1$. $H(e^{j\pi}) = 1/(2-(-1)) = 1/3$. Lowpass.

Impulse invariance $h(n) = Th_a(nT) = (0.001)(1000)e^{-(1000)(0.001)n} = e^{-n}, n \geq 0$.
$H(z) = 1/(1-e^{-1}z^{-1})$ since pole at $-1000 \rightarrow e^{-1000(0.001)} = e^{-1}$.
$H(e^{j0}) = 1/(1-e^{-1}) \approx 1.58$. $H(e^{j\pi}) = 1/(1+e^{-1}) \approx 0.73$. Lowpass.

Bilinear transform $s = \frac{2}{T}\frac{z-1}{z+1} \rightarrow H(z) = 1000/[2000\frac{z-1}{z+1} + 1000] = (z+1)/(3z-1)$.
$H(e^{j0}) = 1$. $H(e^{j\pi}) = 0$. Lowpass; now has zero at -1.

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s=jΩ maps to z=$e^{j\omega}$ $s = \frac{2}{T}\frac{z-1}{z+1} \Leftrightarrow z = (1+sT/2)/(1-sT/2)$. Now set $s = j\Omega$:
$z = (1+j\Omega T/2)/(1-j\Omega T/2) = (1+j\Omega T/2)/(1+j\Omega T/2)^* \rightarrow |z| = 1$.
Imaginary axis (cont. time) mapped to unit circle (discrete time).

Prewarp formula $s = \frac{2}{T}(z-1)/(z+1)|_{z=e^{j\omega}} = \frac{2}{T}(e^{j\omega}-1)/(e^{j\omega}+1)$. Now set $s = j\Omega$:
$j\Omega = \frac{2}{T}[e^{j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})]/[e^{j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})] = j\frac{2}{T}\tan\frac{\omega}{2}$.

Matlab: **[BD,AD]=bilinear(BC,AC,F);** substitutes $s = 2F\frac{z-1}{z+1}$ in $H_a(s)$.
$H_a(s)$ has numerator coefficients **BC** and denominator **AC**.
$H(z)$ has numerator coefficients **BD** and denominator **AD**.
F=1/T (*not* T–watch this!). Need *row* vectors of coefficients.

Matlab: **[B,A]=butter(N,W);** designs *digital* Butterworth filter of order **N** with cutoff frequency **W**$= \omega_o/\pi$.
Use **[B,A]=butter(N,W,'s');** to get *analog* filter.

Goal: Design a digital IIR lowpass filter with sampling rate 3600 Hz.
using: Bilinear transformation (T=2) of Butterworth low-pass filter.
Specs: **DC gain:** 0 dB. **Cutoff:** 600 Hz. **Gain:** -86 dB at 1200 Hz.

Cutoff: **Sampling:** $\omega_c = 2\pi\frac{600}{3600} = \frac{\pi}{3}$. **Bilinear:** $\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{1}{\sqrt{3}}$.
Gain: **Sampling:** $\omega_g = 2\pi\frac{1200}{3600} = \frac{2\pi}{3}$. **Bilinear:** $\Omega_g = \frac{2}{T}\tan\frac{\omega_g}{2} = \sqrt{3}$.

Specs: *Analog* Butterworth filter drops 86 dB in frequency ratio $\frac{\sqrt{3}}{1/\sqrt{3}} = 3$.
Order: *Analog* Butterworth filter order=(86 dB)/($20\log_{10}3$) = 9 (rounded).

Poles: *Analog* Butterworth poles: $\omega_c \exp[j(\frac{\pi}{2}+\frac{(2k+1)\pi}{2N})]$ for $k = 0\dots N-1$.
Here: 9 poles at $\frac{1}{\sqrt{3}}e^{j\theta}$ for θ=100,120,140,160,180,200,220,240,260 degrees.
Transfer: $H_a(s) = 1/[(s-p_1)(s-p_2)\dots(s-p_9)]$ for 9 poles p_k, to scale factor.

Then: Bilinear transformation: Set $s = \frac{2}{T}\frac{z-1}{z+1} = \frac{z-1}{z+1}$ in the above $H_a(s)$:
Get: $H(z) = 1/[(\frac{z-1}{z+1}-p_1)(\frac{z-1}{z+1}-p_2)\dots(\frac{z-1}{z+1}-p_9)]$ which simplifies to

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$H(z) = (z+1)^9/[((z-1)-p_1(z+1))\dots((z-1)-p_9(z+1))]$ becomes
$H(z) = (z+1)^9/[a_0z^9 + a_1z^8 + \dots + a_9] \rightarrow$ ARMA difference equation

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a_0y(n)+a_1y(n-1)+$\dots$ + a_9y(n-9)=x(n)+9x(n-1)+36x(n-2)+84x(n-3)
+126x(n-4)+126x(n-5)+84x(n-6)+36x(n-7)+9x(n-8)+x(n-9).

Note: Using analog filter to design digital filter to implement analog filter.

Problem #2 of Problem Set #9: Single-Pole Filters

Given: $H_a(s) = \frac{a}{s+a}$. **DC gain**=$\frac{a}{0+a}$=1. **3 dB freq.**=a. **Gain=0** at $\omega \rightarrow \infty$.

Bilinear:(T=2) $H(z) = a/[\frac{z-1}{z+1}+a] = \frac{a(z+1)}{(z-1)+a(z+1)} = \frac{a(z+1)}{(a+1)z+(a-1)}$.

DC gain=$\frac{a(1+1)}{(a+1)(1)+(a-1)} = 1$ at $\omega = 0$ after setting $z = e^{j0} = +1$.
Gain=0=$\frac{a(-1+1)}{(a+1)(-1)+(a-1)} = 0$ at $\omega = \pi$ after setting $z = e^{j\pi} = -1$.
3 dB: $|a/[\frac{e^{j\omega}-1}{e^{j\omega}+1}+a]| = \frac{1}{\sqrt{2}} \rightarrow ja = \frac{e^{j\omega}-1}{e^{j\omega}+1} = j\tan\frac{\omega}{2} \rightarrow \omega = 2\tan^{-1}a$.

Prewarp a: $\Omega = a \Leftrightarrow \omega = 2\tan^{-1}a$. Then have $\Omega = \tan\frac{a}{2} \Leftrightarrow \omega = a$.
Then: $H_a(s) = \frac{\tan(a/2)}{s+\tan(a/2)}$ and $s = \frac{z-1}{z+1} \rightarrow H(z) = \tan(a/2)/[\frac{z-1}{z+1}+\tan(a/2)]$.
$H(e^{j\omega}) = \tan(a/2)/[\frac{e^{j\omega}-1}{e^{j\omega}+1}+\tan(a/2)] = \tan(a/2)/[j\tan(\omega/2)+\tan(a/2)]$.
3 dB: $\omega = a \rightarrow |H(e^{ja})| = |\tan(a/2)/[j\tan(a/2)+\tan(a/2)]| = \frac{1}{\sqrt{2}}$.

Why? IIR digital filter design: analog→digital filter using map $s = f(z)$.
What? (1) Specify $|H(\Omega)|^2$; (2) Specify pole locations; (3) Matlab commands.
Stable: Since cont-time, need poles in left-half-plane (LHP). See p. 682-689.
Cutoff: Each filter passes freqs $\Omega < \Omega_c$ and rejects $\Omega > \Omega_c$. Ω_c =cutoff freq.

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1. **Butterworth:** $|H(\Omega)|^2 = 1/[1 + (\frac{\Omega}{\Omega_c})^{2N}]$.

Poles: $\Omega_c e^{j\pi(N+2k+1)/2N}, k = 0, 1 \ldots N-1$. *In LHP* on circle of radius Ω_c.
Why? Maximally flat in passband: $\frac{d^k}{d\Omega^k}|H(\Omega)|^2|_{\Omega=0} = 0, k = 1 \ldots 2N-1$.
But: $|H(\Omega)|$ monotonically decreases with Ω →not a very sharp cutoff.
Matlab: **butter(N,W)**→ *digital* N^{th}-order Butterworth filter with cutoff $W\pi$

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2. **Chebyschev:** $|H(\Omega)|^2 = 1/[1 + \epsilon^2 C_N^2(\frac{\Omega}{\Omega_c})]$.

Polynomial: $C_n(x) = \cos(n \cos^{-1} x) =$ *Chebyschev* polynomial of order n.
Recursion: $C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x)$. $C_0(x) = 1, C_1(x) = x, C_2(x) = 2x^2 - 1$

ϵ: Ripple in passband: $1 \geq |H(\Omega)|^2 \geq 1/(1+\epsilon^2)$. $C_3(x) = 4x^3 - 3x$.
Design: Allowable ripple→ ϵ. Cutoff freq→ Ω_c. Then stopband ripple→ N.

Poles: On ellipse *in LHP* specified from Butterworth filter pole locations by:
Axes: Semimajor: $r_1 = \frac{\Omega_c}{2}(\beta + \frac{1}{\beta})$; semiminor: $r_2 = \frac{\Omega_c}{2}(\beta - \frac{1}{\beta})$; $\beta = [\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}]^{\frac{1}{N}}$.
Poles: $r_2 \cos[\pi(N+2k+1)/2N] + jr_1 \sin[\pi(N+2k+1)/2N], k = 0 \ldots N-1$.

Why? Chebyschev polynomials: minimum bounded variation for $|x| < 1$.
Matlab: **cheby1(N,R,W)**→ *digital* N^{th} Chebyschev with passband ripple R.

Type I: Equiripple in passband, monotonic in stopband. Done above.
Type II: Equiripple in stopband, monotonic in passband. Done below.
Replace both $\frac{\Omega}{\Omega_c}$ and $\epsilon^2 C_N^2(\cdot)$ in Type I with their reciprocals.

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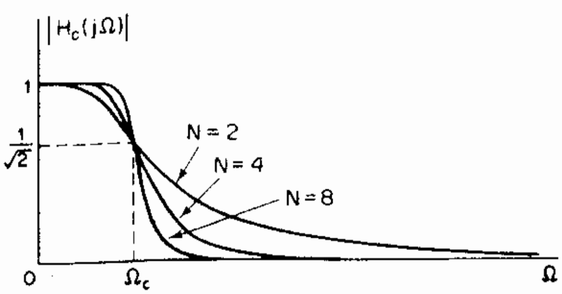
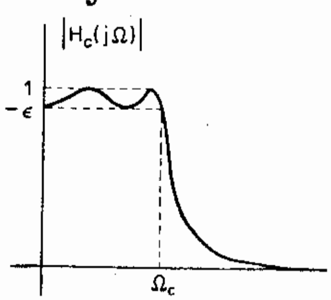
3. **Elliptic:** $|H(\Omega)|^2 = 1/[1 + \epsilon^2 U_N^2(\frac{\Omega}{\Omega_c})]$.

Func: $U_n(x)$ =Jacobian elliptic function (don't ask).
Why? Ripple in both passband and stopband→sharpest transition, given N.
Matlab: **ellip(N,R1,R2,W)**→ *digital* N^{th}-order elliptic with:
ripple: passband ripple R1, stopband ripple R2, cutoff freq. $W\pi$.

Butterworth



Chebyschev I



Elliptic