EECS 451 THE DISCRETE TIME FOURIER SERIES (DTFS)

Cont. Time Fourier Series Let $x(t) = x(t+T)$ be periodic with period=T in **continuous** time.
Then $x(t)$ can be expanded in the *continuous-time* Fourier series
$x(t) = X_0 + X_1 e^{j\frac{2\pi}{T}t} + X_2 e^{j\frac{4\pi}{T}t} + \ldots + X_{-1}e^{-j\frac{2\pi}{T}t} + X_{-2}e^{-j\frac{4\pi}{T}t} + \ldots$
where $X_k = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)e^{-j\frac{2\pi}{T}kt}dt$ for integers k and any time t_o.

Note: **Conjugate symmetry:** $x(t)$ real $\Leftrightarrow X_{-k} = X_k^*$ for integers k.

Discrete Time Fourier Series Let $x[n] = x[n+N]$ be periodic with period=N in **discrete** time.
Then $x[n]$ can be expanded in the *discrete-time* Fourier series
$x[n] = X_0 + X_1 e^{j\frac{2\pi}{N}n} + X_2 e^{j\frac{4\pi}{N}n} + \ldots + X_{N-1}e^{j\frac{(N-1)2\pi}{N}n}$
where $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$=DFT for $k = 0\ldots N-1$.

Note: **Conjugate symmetry:** $x[n]$ real $\Leftrightarrow X_{N-k} = X_k^*$ for integers k.

EX: $x[n] = \{\ldots 12, 6, 4, 6, \underline{12}, 6, 4, 6, 12, 6, 4, 6 \ldots\}$. Periodic; period $N = 4$.

DTFS: $X_0 = \frac{1}{4}(x[0] + (+1)x[1] + (+1)x[2] + (+1)x[3]) = \frac{1}{4}(12+6+4+6) = 7.$

DTFS: $X_1 = \frac{1}{4}(x[0] + (-j)x[1] + (-1)x[2] + (+j)x[3]) = \frac{1}{4}(12-6j-4+6j) = 2.$

DTFS: $X_2 = \frac{1}{4}(x[0] + (-1)x[1] + (+1)x[2] + (-1)x[3]) = \frac{1}{4}(12-6+4-6) = 1.$

DTFS: $X_3 = \frac{1}{4}(x[0] + (+j)x[1] + (-1)x[2] + (-j)x[3]) = \frac{1}{4}(12+6j-4-6j) = 2.$

Note: $X_3 = X_{4-1} = X_1^* = 2^* = 2$ (although both X_3 and X_1 are real here).

Note: $x[n]$ is a real and even function $\Leftrightarrow X_k$ is a real and even function.

Then: $x[n] = 7 + 2e^{j\frac{\pi}{2}n} + 1e^{j\pi n} + 2e^{j\frac{3\pi}{2}n}$ **(complex exponential form)**

Or: $x[n] = 7 + 4\cos(\frac{\pi}{2}n) + 1\cos(\pi n)$ **(trigonometric form)**

since: $e^{j\frac{3\pi}{2}n} = e^{-j\frac{\pi}{2}n}$ (try it) and $e^{j\pi n} = \cos(\pi n) = (-1)^n$.

Power: Time domain: Average power=$\frac{1}{4}(12^2 + 6^2 + 4^2 + 6^2) \quad = 58.$

Parseval: Freq. domain: Average power=$(|7|^2 + |2|^2 + |1|^2 + |2|^2) = 58.$

So? Compute average power in either time domain or frequency domain.

So? Consider $x[n] \to \boxed{LTI} \to y[n]$ where input $x[n] = \{\ldots 12, 6, 4, 6 \ldots\}$

and Linear Time-Invariant (LTI) system is $y[n] - 3y[n-1] = 3x[n] + 3x[n-1]$.

Then: Frequency response function=$H(e^{j\omega}) = 3\frac{e^{j\omega}-1}{e^{j\omega}-3}$ (Huh? stay tuned)

Then: $H(e^{j0}) = 3\frac{+1-1}{+1-3} = 0; \quad H(e^{j\pi/2}) = 3\frac{+j-1}{+j-3} = 1.341e^{-j0.46}$

and: $H(e^{j\pi}) = 3\frac{-1-1}{-1-3} = \frac{3}{2}; \quad H(e^{j3\pi/2}) = 3\frac{-j-1}{-j-3} = 1.341e^{j0.46}$

Then: $y[n] = (0)7 + 1.341e^{-j0.46}2e^{jn\pi/2} + \frac{3}{2}1e^{jn\pi} + 1.341e^{j0.46}2e^{jn3\pi/2}$

and: $y[n] = 7(0) + 4(1.341)\cos(\frac{\pi}{2}n - 0.46) + 1(\frac{3}{2})\cos(\pi n)$ which becomes

$y[n] = 5.366\cos(\frac{\pi}{2}n - 0.46) + 1.5\cos(\pi n)$. Note DC term filtered out.

EXAMPLES OF DTFS PROPERTIES

Given: $x[n]$ is a discrete-time signal with period N: $x[n] = x[n+N]$ for all n.

DTFS: $x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$ where $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$

1. $X_0 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$=DC value=mean value of periodic signal $x[n]$.

2. Negative frequencies are second half of $\{X_k\}$: Use $X_{-k} = X_{N-k}$.

3. Matlab's **fftshift** shifts DC to the center, from the left end of plot. This makes conjugate symmetry $X_{-k} = X_{N-k} = X_k^*$ easier to see.

4. Matlab: `fft(X,N)/N` computes DTFS coefficients if `X` is one period.

EX #1: DTFS$\{\underline{1},0,0,0,0,0,0,0\} = \frac{1}{8}\{1,1,1,1,1,1,1,1\}$. **Impulse** in time.

EX #2: DTFS$\{\underline{0},0,1,0,0,0,0,0\} = \frac{1}{8}\{1,-j,-1,j,1,-j,-1,j\}$. **Delayed** $\delta[n]$.

EX #3: DTFS$\{\underline{1},1,1,1,1,1,1,1\} = \{1,0,0,0,0,0,0,0\}$. **Constant** in time.

EX #4: DTFS$\{\underline{1},1,2,1,1,1,1,1\} = \frac{1}{8}\{9,-j,-1,j,1,-j,-1,j\}$. Is **linear**.

Parseval: Average power $= \frac{11}{8} = \frac{1}{8}(1^2+2^2+1^2+1^2+1^2+1^2+1^2+1^2)$

$= (\frac{1}{8})^2(9^2+|-j|^2+|-1|^2+|j|^2+|1|^2+|-j|^2+|-1|^2+|j|^2)$.

EX #5: DTFS$\{\cos(2\pi\frac{M}{N}n+\theta)\} = \frac{1}{2}e^{j\theta}\delta[k-M] + \frac{1}{2}e^{-j\theta}\delta[k-(N-M)]$.

Note: This only works for *periodic* discrete-time sinusoids: $\omega_o = 2\pi\frac{M}{N}$.

EX: DTFS$\{\underline{24},8,12,16\} = \{15, 3+2j, 3, 3-2j\}$ (1 period of $x[n]$ and X_k).

$\rightarrow$ $x[n] = (15)e^{j0n} + (3+2j)e^{j(\pi/2)n} + (03)e^{j\pi n} + (3-2j)e^{j(3\pi/2)n}$.

1. DTFS$\{\underline{24},16,12,8\} = \{15,3-2j,3,3+2j\}$. **Reversal:** $x[-n] \rightarrow X_k^*$.

Huh? $x[+n] = \{\ldots 24,8,12,16,\underline{24},8,12,16,24,8,12,16\ldots\} \rightarrow$
$x[-n] = \{\ldots 24,16,12,8,\underline{24},16,12,8,24,16,12,8\ldots\}$.

2. DTFS$\{\underline{12},16,24,8\} = \{15,-3-2j,3,2j-3\}$. **Delay:** x[n-D]$\rightarrow X_k e^{\frac{-j2\pi kD}{N}}$.

Huh? $x[n-2] = \{\ldots 12,16,24,8,\underline{12},16,24,8,12,16,24,8\ldots\}$.

3. DTFS$\{\underline{24},-8,12,-16\} = \{3,3-2j,15,3+2j\}$. $x[n]e^{\frac{j2\pi nF}{N}} \rightarrow X_{k-F}$

Huh? "Modulate" signal means *shift* its spectrum by some frequency F.

4. DTFS$\{\underline{24},0,8,0,12,0,16,0\} = \frac{1}{2}\{15,3+2j,3,3-2j,15,3+2j,3,3-2j\}$.

Huh? Interpolate with zeros→repeat and halve DFT of lower order.

5. DTFS$\{\underline{24},8,12,16,24,8,12,16\} = \{15,0,3+2j,0,3,0,3-2j,0\}$.

Huh? Repeat in time→interpolate with zeros in frequency domain.

CONCEPTS BEHIND DISCRETE TIME FOURIER SERIES

Given: $x[n]$ is a discrete-time signal with period N: $x[n] = x[n+N]$ for all n.

DTFS: $x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$ where $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$.

- Fastest-oscillating discrete-time sinusoid: $\omega = \pi \rightarrow \cos(\pi n) = (-1)^n$.

$\rightarrow$ Fourier series of discrete-time periodic signal has **finite** number of terms, with frequencies

$$\{0, \tfrac{2\pi}{N}, 2\tfrac{2\pi}{N}, 3\tfrac{2\pi}{N} \ldots (N-1)\tfrac{2\pi}{N}\} \Leftrightarrow \{0, \pm\tfrac{2\pi}{N}, \pm 2\tfrac{2\pi}{N} \ldots \pm \tfrac{N-1}{2}\tfrac{2\pi}{N}, [\pi?]\}.$$

Huh? If N even, the component with the highest frequency is $\omega = \pi$.
If N odd, the component with the highest frequency is $\omega = \frac{N-1}{N}\pi$.

- If $x[n]$ is real, then $X_{N-k} = X_k^*$ (conjugate symmetry).
- $X_0 = \frac{1}{N}(x[0] + x[1] + \ldots + x[N-1])$=mean value of $x[n]$.
- If N is even, $X_{N/2} = \frac{1}{N}(x[0] - x[1] + x[2] - x[3] + \ldots - x[N-1])$.

SIMPLE EXAMPLE WITH N=4:

Given: $x[n] = \{\ldots 24, 8, 12, 16, \underline{24}, 8, 12, 16, 24, 8, 12, 16 \ldots\}$. Period=N=4.
Goal: Compute DTFS=Fourier series expansion of discrete-time periodic $x[n]$.

- NOTE: $e^{-j\frac{2\pi}{4}1} = -j; e^{-j\frac{2\pi}{4}2} = -1; e^{-j\frac{2\pi}{4}3} = +j$.

1. $X_0 = \frac{1}{4}(24 + 8 + 12 + 16) = 15$. Note this is real.
2. $X_2 = \frac{1}{4}(24 - 8 + 12 - 16) = 03$. Note this is real.
3. $X_1 = \frac{1}{4}(24 + 8(-j) + 12(-1) + 16(+j)) = 3 + 2j$.
4. $X_3 = \frac{1}{4}(24 + 8(+j) + 12(-1) + 16(-j)) = 3 - 2j = X_1^*$.

Then: $x[n] = (15)e^{j0n} + (3+2j)e^{j\frac{2\pi}{4}n} + (03)e^{j\frac{2\pi}{4}2n} + (3-2j)e^{j\frac{2\pi}{4}3n}$.

Line spectrum is **periodic** with components at: $\{0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi \ldots\}$.

Using: $3 + 2j = 3.6e^{j33.7^o}; e^{j\pi n} = \cos(\pi n); e^{j\frac{2\pi}{4}3n} = e^{-j\frac{2\pi}{4}n}$, simplifies to:

$x[n] = 15 + 7.2\cos(\frac{\pi}{2}n + 33.7^o) + 3\cos(\pi n)$. Don't double at $\omega = 0, \pi$.

PARSEVAL'S THEOREM: POWER IS CONSERVED

Power: $\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X_k|^2$=average power of periodic $x[n]$.

Time: $15^2 + |3+2j|^2 + 3^2 + |3-2j|^2 = 260$ since $|3+2j|^2 = 13$.
Freq: $\frac{1}{4}(24^2 + 8^2 + 12^2 + 16^2) = 260$. They are equal!

EXAMPLE OF DISCRETE-TIME FOURIER SERIES (DTFS):

What? Like continuous time, except *finite* #terms→*exact* representation.

Below: $x[n] = c_1\cos(\omega_o n) + c_2\cos(2\omega_o n) + \ldots + c_8\cos(8\omega_o n)$=even function

where: $\omega_o = \frac{2\pi}{N} = \frac{2\pi}{17}$ and $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} = \frac{1}{17}\frac{\sin(9\pi k/17)}{\sin(\pi k/17)}$.