Def: $y(n) = h(n) \circledcirc u(n) = \sum_{i=0}^{N-1} h(i)(u(n-i))_N \Leftrightarrow Y_k = X_k U_k$.
where: $(x(n))_N$ ⇔N-point periodic extension of $x(n)$. "Cyclic"="circular."
Order: "N-point" or "order N"⇔ $y(n), h(n), u(n)$ *all* have length N.

How to Compute Circular Convolutions:

Goal: Compute $\{1,2,3,4\} \circledcirc \{0,1,2,3\}$ in three different ways:
Ref: Phillips and Parr, *Signals, Systems and Transforms*, p.576-580.

#1. Take only *one* cycle of the two cycles shown for each:

$y(0) = \substack{\{1,2,3,4,1,2,3,4\} \\ \{0,3,2,1,0,3,2,1\}} \rightarrow (1)(0)+(2)(3)+(3)(2)+(4)(1) = 16.$

$y(1) = \substack{\{1,2,3,4,1,2,3,4\} \\ \{1,0,3,2,1,0,3,2\}} \rightarrow (1)(1)+(2)(0)+(3)(3)+(4)(2) = 18.$

$y(2) = \substack{\{1,2,3,4,1,2,3,4\} \\ \{2,1,0,3,2,1,0,3\}} \rightarrow (1)(2)+(2)(1)+(3)(0)+(4)(3) = 16.$

$y(3) = \substack{\{1,2,3,4,1,2,3,4\} \\ \{3,2,1,0,3,2,1,0\}} \rightarrow (1)(3)+(2)(2)+(3)(1)+(4)(0) = 10.$

#2. Compute the *linear* convolution and then *alias* it:
Linear: $\{1,2,3,4\} * \{0,1,2,3\} = \{0,1,4,10,16,17,12\}$ (mult. z-transforms).
Alias: $\rightarrow \{0+16, 1+17, 4+12, 10\} = \{16,18,16,10\}$ checks.

#3. Compute 4-point DFTs, multiply, compute 4-point inverse DFT:
H_k: $DFT\{1,2,3,4\} = \{10, -2+j2, -2, -2-j2\}$. Confirm this!
U_k: $DFT\{0,1,2,3\} = \{06, -2+j2, -2, -2-j2\}$. Confirm this!
H_kU_k: $\{(10)(6), (-2+j2)(-2+j2), (-2)(-2), (-2-j2)(-2-j2)\} = \{60, -j8, 4, j8\}$.
$y(n)$: $y(n) = DFT^{-1}\{60, -j8, 4, j8\} = \{16,18,16,10\}$ checks.

Using Cyclic Convs and DFTs to Compute Linear Convs:

0-pad: $\{1,2,3,4,0,0,0\} \circledcirc \{0,1,2,3,0,0,0\} = \{0,1,4,10,16,17,12\}$.
Note: Linear conv. of two 4-point→ $4+4-1 = 7$-point sequence.

Long input u(n):
Often input signal $u(n)$ is much longer than filter $h(n)$.
Chop up $u(n)$ into *segments* $u_i(n)$ and compute $h(n) * u_i(n)$.
Use *Overlap-save* or *overlap-add* methods (see text p.430-433).

Compute quickly by multiplying 7-point DFTs, then inverse DFT:

EECS 451 COMPUTING CONTINUOUS-TIME FOURIER TRANSFORMS USING THE DFT

Goal: Compute *numerically* $X(f) = \int x(t)e^{-j2\pi ft}dt; x(t) = \int X(f)e^{j2\pi ft}df$.
Note $X(f) = X(\frac{\omega}{2\pi})$ and $df = \frac{d\omega}{2\pi}$ (note missing 2π in inverse).

Assume: $x(t)$ *time-limited* to $0 < t < T$ (so that $x(t) = 0$ for other values of t).
What if $x(t)$ is time-limited to another interval $T_i < t < T_f$?
Then use $T = T_f - T_i$ and multiply $X(f)$ by $e^{-j2\pi fT_i}$ afterwards.

Assume: $X(f)$ *band-limited* to $-B/2 < f < B/2$ ($X(f) = 0$ for $|f| > B/2$).
Why $B/2$ instead of B? Saves factor 2 throughout below; symmetry.

Sample t: $t = n\Delta_t$ for $0 \leq n \leq N-1$. Nyquist sampling$\rightarrow$ $\Delta_t = 1/B$.
Sample f: $f = k\Delta_f$ for $0 \leq k \leq N-1$. Nyquist sampling$\rightarrow$ $\Delta_f = 1/T$.

Approx: $X(f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi fn\Delta_t}\Delta_t$ (using rectangle rule).
Sample f: $X(k\Delta_f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi nk\Delta_t\Delta_f}\Delta_t$.
Inverse: Repeating above, $x(n) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi nk\Delta_t\Delta_f}\Delta_f$

DFT: Let $X_k = X(k\Delta_f)/\Delta_t = X(k\Delta_f)B$ and $\Delta_t\Delta_f = 1/N$ above.
Then: $X_k \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi nk/N}$; $\quad x(n) \approx \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$ DFT!

FFT: Use FFT of order $N = BT$ to compute DFT of order $N = BT$.
where: $\Delta_t\Delta_f = 1/N$ and $\Delta_t = 1/B$ and $\Delta_f = 1/T$ all$\rightarrow$ $BT = N$.

Formulae: $BT = N$; $\Delta_t\Delta_f = 1/N$; $\Delta_t = 1/B$; $\Delta_f = 1/T$; $X_k = X(k\Delta_f)B$.
Remember: $X(f) = 0$ for $|f| > B/2$, *not* $|f| > B$!

Example: $\mathcal{F}\{e^{-|t|}\} = 2/(\omega^2 + 1) = 2/(4\pi^2 f^2 + 1)$. Recall $\omega = 2\pi f$.
Limits: About $-6 < t < 6 \rightarrow T = 12$ and $-8 < f < 8 \rightarrow B = 16$ (*not* 8!).
Values: $N = BT = (16)(12) = 192$; $\Delta_t = 1/B = 1/16$; $\Delta_f = 1/T = 1/12$.
FFT: 192-point FFT of $x(t)$ sampled 1/16 to get $16X(f)$ sampled 1/12.
