# BASIC SYSTEM PROPERTIES

**What:** Input $x[n] \rightarrow \overline{\underline{|\text{SYSTEM}|}} \rightarrow y[n]$ output.

**Why:** **Design** the system to **filter** input $x[n]$.

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**DEF:** A system is **LINEAR** if these two properties hold:

1. **Scaling:** If $x[n] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow y[n]$, then $ax[n] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow ay[n]$

**for:** any **constant** $a$. NOT true if $a$ varies with time (i.e., $a[n]$).

2. **Superposition:** If $x_1[n] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow y_1[n]$ and $x_2[n] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow y_2[n]$,

**Then:** $(ax_1[n] + bx_2[n]) \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow (ay_1[n] + by_2[n])$

**for:** any **constants** $a, b$. NOT true if $a$ or $b$ vary with time (i.e., $a[n], b[n]$).

**EX:** $y[n] = 3x[n-2]$; $y[n] = x[n+1] - nx[n] + 2x[n-1]$; $y[n] = \sin(n)x[n]$.

**NOT:** $y[n] = x^2[n]$; $y[n] = \sin(x[n])$; $y[n] = |x[n]|$; $y[n] = x[n]/x[n-1]$.

**NOT:** $y[n] = x[n] + 1$ (try it). This is called an **affine** system.

**HOW:** If any nonlinear function of $x[n]$, not linear. Nonlinear of just $n$ OK.

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**DEF:** A system is **TIME-INVARIANT** if this property holds:

If $x[n] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow y[n]$, then $x[n-N] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow y[n-N]$

**for:** any integer time delay $N$. NOT true if $N$ varies with time (e.g., $N(n)$).

**EX:** $y[n] = 3x[n-2]$; $y[n] = \sin(x[n])$; $y[n] = x[n]/x[n-1]$.

**NOT:** $y[n] = nx[n]$; $y[n] = x[n^2]$; $y[n] = x[2n]$; $y[n] = x[-n]$.

**HOW:** If $n$ appears anywhere other than in $x[n]$, not time-invariant. Else OK.

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**DEF:** A system is **CAUSAL** if it has this form for some function $F(\cdot)$:
$y[n] = F(x[n], x[n-1], x[n-2] \ldots)$ (present and past input only).

**Note:** Physical systems must be causal. But DSP filters need not be causal!

**DEF:** A system is **MEMORYLESS** if $y[n] = F(x[n])$ (present input only).

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**DEF:** A system is **(BIBO) STABLE** iff: Let $x[n] \rightarrow \overline{\underline{|\text{SYSTEM}|}} \rightarrow y[n]$.

If $|x[n]| < M$ for some constant $M$, then $|y[n]| < N$ for some $N$.

**i.e.:** "Every bounded input (BI) produces a bounded output (BO)."

**HOW:** BIBO stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < L$ for some constant $L$

**where:** Impulse $\delta[n] \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow h[n]$=impulse response.

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**EX:** A time-invariant system is observed to have these two responses:
$\{\underline{0}, 0, 3\} \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow \{\underline{0}, 1, 0, 2\}$ and $\{\underline{0}, 0, 0, 1\} \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow \{1, \underline{2}, 1\}$.

**Prove:** The system is nonlinear.

**Proof:** By contradiction. Suppose the system is linear. But then:
$\{\underline{0}, 0, 0, 1\} \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow \{1, \underline{2}, 1\}$ implies $\{\underline{0}, 0, 3\} \rightarrow \overline{\underline{|\text{SYS}|}} \rightarrow \{3, 6, \underline{3}\}$
since we know it is time-invariant. Then $\{\underline{0}, 0, 3\}$ produces two outputs!

## CONVOLUTION AND IMPULSE RESPONSE

$x[n] = \{3, \underline{1}, 4, 6\} \Leftrightarrow x[n] = 3\delta[n+1] + 1\delta[n] + 4\delta[n-1] + 6\delta[n-2]$.

**Note:** $x[n] = \sum_i x[i]\delta[n-i] = x[n] * \delta[n]$ (sifting property of impulse).

**Delay:** $x[n-D]$ is $x[n]$ shifted *right* (later) if $D > 0$; *left* (earlier) if $D < 0$.

**Fold:** $x[-n]$ is $x[n]$ flipped/folded/reversed around $n = 0$.

**Both:** $x[N-n]$ is $x[-n]$ shifted *right* if $N > 0$ (since $x[0]$ is now at $n = N$).

## FOR LINEAR TIME-INVARIANT (LTI) SYSTEMS:

1. $\delta[n] \to \overline{|LTI|} \to h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \to \overline{|LTI|} \to h[n-i]$ **Time invariant:** delay by $i$.
3. $x[i]\delta[n-i] \to \overline{|LTI|} \to x[i]h[n-i]$ **Linear:** scale by $x[i]$.
4. $\sum_i x[i]\delta[n-i] \to \overline{|LTI|} \to \sum_i x[i]h[n-i]$ **Linear:** superposition.
5. $x[n] \to \overline{|LTI|} \to y[n] = \sum_i x[i]h[n-i] = h[n] * x[n]$ **Convolution**.
   Input $x[n]$ into LTI system with no initial stored energy→output $y[n]$.

## PROPERTIES OF DISCRETE CONVOLUTION

1. $y[n] = h[n] * x[n] = x[n] * h[n] = \sum_i h[i]x[n-i] = \sum h[n-i]x[i]$.
2. $h[n], x[n]$ both causal ($h[n] = 0$ for $n < 0$ and $x[n] = 0$ for $n < 0$)
   $\to y[n] = \sum_{i=0}^{n} h[i]x[n-i] = \sum_{i=0}^{n} h[n-i]x[i]$ also causal.

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3. Suppose $h[n] \neq 0$ *only for* $0 \leq n \leq L$ ($h[n]$ has length $L+1$).
   Suppose $x[n] \neq 0$ *only for* $0 \leq n \leq M$ ($x[n]$ has length $M+1$).
   Then $y[n] \neq 0$ only for $0 \leq n \leq L+M$ ($y[n]$ has length $L+M+1$).

**Note:** Length[$y[n]$]=Length[$h[n]$]+Length[$x[n]$]-1.

**Note:** $y[0] = h[0]x[0]$; $y[L+M] = h[L]x[M]$; $x[n] * \delta[n-D] = x[n-D]$.

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4. $x[n] \to \overline{|h_1[n]|} \to \overline{|h_2[n]|} \to y[n]$ (cascade connection)
   Equivalent to: $x[n] \to \overline{|h_1[n] * h_2[n]|} \to y[n]$.

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5. $x[n] \to$ ↗ $\to \overline{|h_1[n]|} \to$ ↘ / ↘ $\to \overline{|h_2[n]|} \to$ ↗ $\bigoplus \to y[n]$ (parallel connection)
   Equivalent to: $x[n] \to \overline{|h_1[n] + h_2[n]|} \to y[n]$.

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**MA:** $y[n] = b_0x[n] + b_1x[n-1] + \ldots + b_qx[n-q]$ (Moving Average)

**Huh?** Present output=weighted average of $q$ *most recent* inputs.

**Note:** Equivalent to $y[n] = b[n] * x[n]$ where $b[k] = b_k, 0 \leq k \leq q$.

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**FIR:** Finite Impulse Response $\Leftrightarrow h[n]$ has finite *duration*.

**EX:** Any MA system is also an FIR system, and vice-versa.

**IIR:** Infinite Impulse Response $\Leftrightarrow h[n]$ not finite duration.

**EX:** $h[n] = a^n u[n] = a^n$ for $n \geq 0$ and $|a| < 1$ is stable and IIR.

## RECURSIVE COMPUTATION OF IMPULSE RESPONSE

**Goal:** Compute impulse response $h[n]$ of system $y[n]-\frac{1}{2}y[n-1]=3x[n]$.

**Sol'n:** Compute recursively $h[n]-\frac{1}{2}h[n-1]=3\delta[n]=0$ if $n>0$.

**n=0:** $h[0]-\frac{1}{2}h[-1]=3\delta[0]\rightarrow h[0]-\frac{1}{2}(0)=3(1)\rightarrow h[0]=3$.

**n=1:** $h[1]-\frac{1}{2}h[0]=3\delta[1]\rightarrow h[1]-\frac{1}{2}(3)=3(0)\rightarrow\ h[0]=\frac{3}{2}$.

**n=2:** $h[2]-\frac{1}{2}h[1]=3\delta[2]\rightarrow h[2]-\frac{1}{2}(\frac{3}{2})=3(0)\rightarrow h[0]=\frac{3}{4}$.

**n=3:** $h[3]-\frac{1}{2}h[2]=3\delta[3]\rightarrow h[3]-\frac{1}{2}(\frac{3}{4})=3(0)\rightarrow h[0]=\frac{3}{8}$.

$h[n]=3(\frac{1}{2})^n u[n]=3(\frac{1}{2})^n$ for $n\geq 0$. *Geometric* signal.

## BIBO (BOUNDED INPUT→BOUNDED OUTPUT) STABILITY

**Goal:** Determine whether an LTI system is BIBO stable from its $h[n]$.

**EX #1:** $h[n]=\{2,3,-4\}\rightarrow\sum|h[n]|=|2|+|3|+|-4|=9<\infty\rightarrow$ BIBO stable.

**EX #2:** $h[n]=(-\frac{1}{2})^n u[n]\rightarrow\sum|h[n]|=\sum|-\frac{1}{2}|^n=\frac{1}{1-0.5}<\infty\rightarrow$ BIBO stable.

**EX #3:** $h[n]=\frac{(-1)^n}{n+1}u[n]\rightarrow\sum|h[n]|=\sum\frac{1}{n+1}u[n]\rightarrow\infty\rightarrow$ NOT BIBO stable.

**Note:** $\sum\frac{(-1)^n}{n+1}u[n]=\log 2$ but $\sum|\frac{(-1)^n}{n+1}|u[n]=\sum\frac{1}{n+1}u[n]\rightarrow\infty$
so *absolute* summability vs. summability matters for BIBO stability!

## CONVOLUTION OF TWO FINITE SIGNALS

**Goal:** Compute $\{\underline{1},2,3\}*\{\underline{4},5,6,7\}=\{4,13,28,34,32,21\}$.

$y[0]=h[0]x[0]=(1)(4)=04$.

$y[1]=h[1]x[0]+h[0]x[1]=(2)(4)+(1)(5)=13$.

$y[2]=h[2]x[0]+h[1]x[1]+h[0]x[2]=(3)(4)+(2)(5)+(1)(6)=28$.

$y[3]=h[2]x[1]+h[1]x[2]+h[0]x[3]=(3)(5)+(2)(6)+(1)(7)=34$.

$y[4]=h[2]x[2]+h[1]x[3]=(3)(6)+(2)(7)=32$.

$y[5]=h[2]x[3]=(3)(7)=21$.

**Note:** Length $y[n]$=Length $h[n]$+Length $x[n]-1=3+4-1=6$.

## CONVOLUTION OF FINITE AND INFINITE SIGNALS

**Goal:** Compute $\{\underline{2},-1,3\}*(\frac{1}{2})^n u[n]=2\delta[n]+3(\frac{1}{2})^{n-2}u[n-2]$.

**Sol'n:** $\{\underline{2},-1,3\}*(\frac{1}{2})^n u[n]=(2\delta[n]-1\delta[n-1]+3\delta[n-2])*(\frac{1}{2})^n u[n]=$
$2(\frac{1}{2})^n u[n]-1(\frac{1}{2})^{n-1}u[n-1]+3(\frac{1}{2})^{n-2}u[n-2]=2\delta[n]+3(\frac{1}{2})^{n-2}u[n-2]$.

**Note:** $\{\underline{2},-1\}*(\frac{1}{2})^n u[n]=2\delta[n]$. So $x[n]\rightarrow\boxed{(\frac{1}{2})^n u[n]}\rightarrow\boxed{\{2,-1\}}\rightarrow 2x[n]$.

**MA:** $y(n) = b_0 x(n) + b_1 x(n-1) + \ldots + b_q x(n-q)$ (Moving Average)
**Huh?** Present output=weighted average of $q$ *most recent* inputs.
**Note:** Equivalent to $y(n) = b(n) * x(n)$ where $b(k) = b_k, 0 \leq k \leq q$.
**Why?** So why bother? Because now can have *initial conditions*.

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**AR:** $y(n) + a_1 y(n-1) + \ldots + a_p y(n-p) = x(n)$ (AutoRegression)
**Huh?** Present output=weighted sum of $p$ *most recent* **outputs**.
**Note:** Compute $y(n)$ *recursively* from its $p$ most recent values.

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**ARMA:** $$\underbrace{\sum_{i=0}^{p} a_i y(n-i)}_{\text{AUTOREGRESSIVE}} = \underbrace{\sum_{i=0}^{q} b_i x(n-i)}_{\text{MOVING AVERAGE}}$$ (Combine AR and MA→ARMA).

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## PROPERTIES OF DIFFERENCE EQUATIONS

1. Clearly represent LTI system. Now allow initial conditions.
2. MA model⇔FIR (Finite Impulse Response) filter:
   For MA model, $h(n)$ has finite length $q+1$.
3. AR model⇔IIR (Infinite Impulse Response) filter:
   For AR model, $h(n)$ does *not* have finite length.
   But can implement using finite set of *coefficients* $a_i$.

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4. Analogous to *differential equation* in continuous time:
   $$\left(\frac{d^p}{dt^p} + a_1 \frac{d^{p-1}}{dt^{p-1}} + \ldots + a_p\right) y(t) = \left(\frac{d^q}{dt^q} + b_1 \frac{d^{q-1}}{dt^{q-1}} + \ldots + b_q\right) x(t).$$

**Note:** Coefficients $a_i$ and $b_i$ are *not directly* analogous here.
**Note:** Time-invariant in both cases since coefficients indpt of time.

5. SKIP Section 2.4.3 in text: 1-sided z-transform is *much* easier.
   READ Section 2.5 in text: signal flow graph implementations.

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## DIFFERENCE EQUATIONS VS. $h(n)$

**Advantages** of Difference Equations:

1. Analogous to *differential equation* in continuous time.
   Often can write them down from *model* of physical system.
2. Can incorporate nonzero initial conditions.
3. AR model may be much more efficient implementation of system.

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**Disadvantages** of Difference Equations:

1. There may not be a difference equation description of system!
   Example: $h(n) = \frac{1}{n}, n > 0$ has no difference equation model.
   Recall: $h(t) = \delta(t-1)$ (delay) has no differential equation model.
2. May only want output $y(N)$ for single time $N$ (e.g., end of year).
   Then convolution $y(N) = \sum h(i) x(N-i)$ more efficient computation.