CONTINUOUS-TIME AND DISCRETE-TIME SINUSOIDS

**DEF:** A sinusoidal signal has the standard form \( x(t) = A \cos(\omega_o t + \phi) \).

**Where:** \( A \geq 0 = \text{amplitude}; \) \( \omega_o \geq 0 = \text{frequency}; \) \( |\phi| \leq \pi = \text{phase} \) (shift).

**Freq:** \( f_o = \omega_o/(2\pi) = \text{circular or cyclic frequency in Hertz} = \text{cycles per second}. \)

**Note:** Sinusoid \( x(t) = A \cos(2\pi f_o t + \phi) \) has cyclic frequency \( f_o \) Hertz.

**Period:** \( T = \frac{1}{f_o} = \frac{2\pi}{\omega_o} \rightarrow x(t) = x(t + T) \) for all \( t \) using \( \cos(x) = \cos(x + 2k\pi) \).

**DC:** DC (constant) signals: \( \omega_o = f_o = 0; \) \( T = \infty \) (low-frequency limit).

\[
\text{RMS}(x) = \sqrt{\frac{1}{T} \int_0^T A^2 \cos^2(\omega_o t + \phi) dt} = \sqrt{\frac{1}{T} \int_0^T A^2}{\left(1 + \cos(2\omega_o t + 2\phi)\right)} dt = \frac{A}{\sqrt{2}}
\]

**Note:** Can convert other forms to this standard form. Some examples:

1. \( x(t) = -3 \cos(7t + 0.1) = 3 \cos(7t \pm \pi + 0.1) \) using \( -\cos(x) = \cos(x \pm \pi) \).
2. \( x(t) = 3 \sin(7t + 0.1) = 3 \cos(7t - \frac{\pi}{2} + 0.1) \) using \( \sin(x) = \cos(x - \frac{\pi}{2}) \).
3. \( x(t) = -3 \cos(-7t + 0.1) = 3 \cos(7t \pm \pi - 0.1) \) using \( \cos(-x) = \cos(x) \).

**x(t):** Continuous-time sinusoids are periodic with period \( T = \frac{1}{f_o} = \frac{2\pi}{\omega_o} \).

**x[n]:** Discrete-time sinusoids are NOT periodic UNLESS \( f_o \) is rational.

\( \omega_o = 2\pi \frac{K}{T} \leftrightarrow x[n] \) has period \( T \), provided \( \frac{K}{T} \) is reduced to lowest terms.

**Why?** \( A \cos(\omega_o n + \phi) = A \cos(\omega_o(n + T) + \phi) \leftrightarrow \omega_o T = 2\pi K \leftrightarrow f_o = \frac{\omega_o}{2\pi} = \frac{K}{T} \).

**EX:** \( \omega_o = 2\pi \rightarrow T = 1; \) \( \omega_o = (2.02)\pi \rightarrow T = 100; \) \( \omega_o = 2 \rightarrow \text{nonperiodic} \).

**Freq.:** \( A \cos(\omega_o n + \phi) = A \cos([\omega_o + 2k\pi]n + \phi) \rightarrow \omega_o \text{ itself} \) is periodic.

As \( \omega_o \) increases from 0 to \( \pi \), oscillation rate increases.

As \( \omega_o \) increases from \( \pi \) to 2\( \pi \), oscillation rate decreases.

**Huh?** Frequency range \( \pi \leq \omega_o \leq 2\pi \leftrightarrow -\pi \leq \omega_o \leq 0 \) since \( \omega_o \) periodic.

**WLOG:** Restrict discrete-time frequencies to range \( -\pi \leq \omega_o \leq \pi \).

**Note:** \( \omega_o = 0 \rightarrow \cos(\omega_o n) = 1; \) \( \omega_o = \pm \pi \rightarrow \cos(\omega_o n) = (-1)^n \).

SUM OF MULTIPLE SINUSOIDS

If: \( x(t) \) has period \( T_x \) and freq. \( f_x \); \( y(t) \) has period \( T_y \) and freq. \( f_y \);

And: \( T_x/T_y \) is a rational number (otherwise \( x(t) + y(t) \) not periodic);

Then: \( x(t) + y(t) \) has period \( T_{x+y} = \text{Least Common Multiple of} \ T_x \text{ and} \ T_y \);

Then: \( x(t) + y(t) \) has freq. \( f_{x+y} = \text{Greatest Common Divisor of} \ f_x \text{ and} \ f_y \).

**Note:** Always works for sinusoids; rarely fails for general periodic signals.

**How?** (1) Reduce \( \frac{T_x}{T_y} = \frac{M}{N} \) to lowest terms. (2) \( T_{x+y} = NT_x = MT_y \).

**Why?** \( \frac{T_x}{T_y} = \frac{M}{N} \rightarrow x(t + T_{x+y}) + y(t + T_{x+y}) = x(t + NT_x) + y(t + MT_y) \).

**EX:** \( x(t) = \cos(0.3\pi t) + \cos(2\pi/3 t) \rightarrow f_o = \frac{3}{20}, \frac{1}{3} \rightarrow f_{x+y} = \frac{1}{60} \rightarrow T_{x+y} = 60 \).
ELEMENTARY SIGNAL OPERATIONS: TRANSLATION

**Value:** $y(t) = x(t) + c$ shifts plot of $x(t)$ up by $c$ if $c > 0$ (down if $c < 0$).
**Value:** $y(t) = cx(t)$ scales vertical axis of plot by factor $c$.

**Time:** $y(t) = x(t-d)$ shifts plot of $x(t)$ right by $d$ if $d > 0$ (left if $d < 0$).
**Time:** $y(t) = x(dt)$ scales horizontal axis of plot by factor $d$.
**Time:** $y(t) = x(-t)$ reverses: flips plot about $t = 0$ axis.

**Note:** For discrete time signals $x[n]$, $d$ must be an integer!

**EX #1:** $y(t) = x(t-2)$ delays $x(t)$ by 2: $y(2) = x(0), y(3) = x(1)$.
**EX #2:** $y(t) = x(2t)$ shrinks $x(t)$ by 2: $y(1) = x(2), y(2) = x(4)$.
**Means:** If $x(t) = \cos(\omega_0 t)$, this doubles frequency & halves period.

**EX #3:** $y(t) = x(-t)$: If $x(t)$ audio, then $y(t)$ is $x(t)$ backwards!
Play Beatles records backwards to find hidden messages? Doppler.

**COMBINING SIGNAL OPERATIONS**

- One of the trickiest topics in EECS 216! Be very careful.
- Two methods are available. Pick one and stick with it.

**Why?** See lecture notes. Reread at least three times.

- **First Method: Shift then Scale**
  1. Put problem into the form $y(t) = x(at - b)$.
  2. Shift $x(t)$ by $b$. $b > 0 \rightarrow$ shift right. $b < 0 \rightarrow$ shift left.
  3. Scale result of #2 in time by $a$. $a > 1 \rightarrow$ compress. $0 < a < 1 \rightarrow$ expand.
     $a < 0 \rightarrow$ time reversal: flip plot around $t = 0$ axis.

- **Second Method: Scale then Shift**
  1. Put problem into the form $y(t) = x(c(t-d))$.
  2. Scale $x(t)$ in $t$ by $c$. $c > 1 \rightarrow$ compress in time. $0 < c < 1 \rightarrow$ expand.
     $c < 0 \rightarrow$ time reversal: flip plot around $t = 0$ axis.
  3. Shift result of #2 in time by $d$. $d > 0 \rightarrow$ shift right. $d < 0 \rightarrow$ shift left.

**EX:** $x(t) = 5\cos(3t+2)$. $y(t) = 4x(2t-1)$. $y(3) = 4x(6-1) = 20\cos(17)$.
**Result:** $y(t) = 20\cos(3(2t-1)+2) = 20\cos(6t-1)$. Frequency doubles.

**DEF:** Linear combination $z(t)$ of $x(t)$ and $y(t)$:
$z(t) = ax(t) + by(t)$ for two constants $a$ and $b$.

**NOT:** $tx(t) + 3y(t)$ is NOT a linear combination of $x(t)$ and $y(t)$.
**EX:** Audio mixing; summing Fourier series (later in EECS 216).

**DEF:** Concatenation $z(t)$ of finite-support $x(t)$ and $y(t)$:
$x(t)$ has support $[0, T_x] \rightarrow z(t) = x(t)$ for $0 \leq t \leq T_x$.
$y(t)$ has support $[0, T_y] \rightarrow z(t) = y(t-T_x)$ for $T_x \leq t \leq (T_x + T_y)$.
$z(t)$ has support $[0, T_x + T_y] \leftrightarrow z(t) = 0$ for $t < 0$ or $t > (T_x + T_y)$. 