LINE SPECTRUM OF SIGNAL=SUM OF SINUSOIDS

**Given:** \( x(t) = A_0 + A_1 \cos(\omega_1 t + \theta_1) + \ldots + A_N \cos(\omega_N t + \theta_N) \).

Sum of \( N \) sinusoids at \( N \) different frequencies.

**Goal:** Represent amplitudes and phases as scaled impulses vs. frequency.

Using \( Re[z] = \frac{1}{2}(z + z^*) \) with \( z = X_k e^{j\omega_k t} \), we have:

\[
x(t) = A_0 + \frac{1}{2} \sum_{k=1}^{N}(X_k e^{j\omega_k t} + X_k^* e^{-j\omega_k t}) \quad \text{where} \quad X_k = A_k e^{j\theta_k}.
\]

1. Can represent a sinusoid at frequency \( \omega_k = 2\pi f_k \) by two complex exponentials at frequencies \( \pm \omega_k \).
2. Note need negative frequencies; watch amplitude factor of two.
3. Note component at negative frequency \( -\omega_k \) is complex conjugate of its counterpart component at positive frequency \( \omega_k \).
4. Plot: impulses=lines in frequencies; hence line spectrum.

**EX:** \( x(t) = 10 + 14 \cos(2t - 1) + 8 \cos(4t + 3) \) has 5 complex components:

- (10 at \( \omega = 0 \));
- \( (7e^{\pm j} \text{ at } \omega = \pm 2) \);
- \( (4e^{\pm j^3} \text{ at } \omega = \pm 4) \) (see plot overleaf).

BEAT SIGNALS (SOUND LIKE THEY ARE BEATING)

**What:** Identity \( \cos(\omega_1 t + \theta_1) + \cos(\omega_2 t + \theta_2) = 2 \cos(\omega_c t + \theta_c) \cos(\omega_\Delta t + \theta_\Delta) \).

where: \( \omega_c = \frac{1}{2}(\omega_1 + \omega_2) \) and \( \omega_\Delta = \frac{1}{2}(\omega_1 - \omega_2) \); \( \theta_c, \theta_\Delta \) are defined similarly.

**Why?** Apply cosine addition formula and \( \omega_1 = \omega_c + \omega_\Delta \) and \( \omega_2 = \omega_c - \omega_\Delta \).

**Easier:** Note that \( \cos(\omega_c t + \theta_c) \) can be written as product

\[
[e^{j(\omega_c t + \theta_c)} + e^{-j(\omega_c t + \theta_c)}][e^{j(\omega_\Delta t + \theta_\Delta)} + e^{-j(\omega_\Delta t + \theta_\Delta)}] = 4 \cos(\omega_1 t + \theta_1) + 2 \cos(\omega_2 t + \theta_2).
\]

Multiply this out→4 terms→ \( 2 \cos(\omega_1 t + \theta_1) + 2 \cos(\omega_2 t + \theta_2) \) (try it!).

**Sounds** \( x(t) \) sounds like a single tone at frequency \( \omega_c \approx \omega_1, \omega_2 \)

like: whose amplitude varies slowly with time at frequency \( \omega_\Delta \).

**EX #1:** Tuning a piano. We wish to tune a piano key to 440 Hz.

**Given:** A tuning fork tuned to A above middle C, which is 440 Hz.

**Given:** Piano mistuned to 441 Hz. Want to change this to 440 Hz.

**Sol’n:** Strike tuning fork and piano key simultaneously and listen.

\[
\rightarrow \text{Hear } \cos(2\pi 440t) + \cos(2\pi 441t) = 2 \cos(2\pi 0.5t) \cos(2\pi 440.5t)
\]

**Sounds** A single tone at 440.5 Hz whose amplitude varies at 0.5 Hz.

like: Actually 1 Hz since amplitude is non-negative. Period=1 second.

**Sol’n:** Adjust piano key until amplitude variation ceases (Period→∞).

**EX #2:** AM radio. WJR (760 kHz) transmits a tone (EBS?) at 5 kHz.

\[
x(t) = 4 \cos(2\pi(760,000)t) \cos(2\pi(5000)t) = 2 \cos(2\pi(755,000)t) + 2 \cos(2\pi(765,000)t)
\]

\[
= e^{j2\pi(765,000)t} + e^{j2\pi(755,000)t} + e^{-j2\pi(765,000)t} + e^{-j2\pi(755,000)t}.
\]

**Line spectrum:** 4 components in two closely-spaced pairs.
\[ 10 + 14\cos(2t-1) + 8\cos(4t+3) \]

**Line Spectrum of Sum**

- \( 7e^{j} \)
- \( 7e^{-j} \)
- \( 4e^{j3} \)
- \( 4e^{-j3} \)

**BEAT**

\[ 4\cos(2t)\cos(0.2t) \]

**Line Spectrum of Beat**

**AM**

\[ 4\cos(0.05t)\cos(4t) \]

**Line Spectrum of AM**