COMPLEX EXPONENTIAL FOURIER SERIES

Given: $x(t)$ is continuous-time *periodic* function: Period $T \rightarrow x(t) = x(t+T)$.

Series: $x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j2\pi kt/T}$; $\quad x_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j2\pi kt/T} dt$.
Discrete in frequency⇔Periodic in time.

Dirichlet (Sufficient) Conditions for Convergence:

Historical Note: (Bracewell p.205) At a meeting of the Paris Academy in 1807: Fourier claimed any periodic function could be expanded in sinusoids. Lagrange stood up and said he was wrong. Led to Riemann integral.

If: Over each period (any interval of length T):

1. $x(t)$ has a **finite number** of discontinuities, maxima, and minima;
2. $x(t)$ is **absolutely integrable**: $\int_{-T/2}^{T/2} |x(t)| dt < \infty$.

Then: $\underset{N\rightarrow\infty}{\text{LIM}} |x(t) - \sum_{k=-N}^{N} x_k e^{j2\pi kt/T}| = 0$ for all t

where: at discontinuities of $x(t)$ at t_i, convergence is to $\frac{1}{2}(x(t_i^+) + x(t_i^-))$.

None: $x(t) = 1/(1-t) \rightarrow$no Fourier series: not absolutely integrable.
None: $x(t) = \sin(1/t) \rightarrow$no Fourier series: ∞ maxima and minima.

Finite energy in one period→Mean-square convergence (MSC)(weaker):
MSC: $\int_{-T/2}^{T/2} |x(t)|^2 dt < \infty \rightarrow \underset{N\rightarrow\infty}{\text{LIM}} \int_{-T/2}^{T/2} |x(t) - \sum_{k=-N}^{N} x_k e^{j2\pi kt/T}|^2 dt = 0$.

PROPERTIES OF FOURIER SERIES

1. Can also use $x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi kt/T) + b_k \sin(2\pi kt/T)$.

   $a_0 = x_0 = \frac{1}{T}\int x(t) dt$ (integrate over 1 period; use everywhere below).

   $a_k = x_k + x_{-k} = 2\,Re[x_k] = \frac{2}{T}\int x(t)\cos(2\pi kt/T) dt$. $\mathbf{x_k = \frac{1}{2}(a_k - jb_k)}$

   $b_k = j(x_k - x_{-k}) = -2\,Im[x_k] = \frac{2}{T}\int x(t)\sin(2\pi kt/T) dt$. Note signs!

2. **Parseval:** Power$=\frac{1}{T}\int |x(t)|^2 dt = \sum_{-\infty}^{\infty} |x_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$.
3. **Even/odd:** $x_o(t) = \sum_{k=1}^{\infty} b_k \sin(2\pi kt/T); x_e(t) = \sum_{k=0}^{\infty} a_k \cos(2\pi kt/T)$.
4. **Orthogonality:** $\int_0^T e^{-j2\pi mt/T} e^{j2\pi nt/T} dt = T\delta(m-n)$. **If:** $m, n \neq 0$:

   $\int_0^T \cos(2\pi \frac{mt}{T})\cos(2\pi \frac{nt}{T}) dt = \int_0^T \sin(2\pi \frac{mt}{T})\sin(2\pi \frac{nt}{T}) dt = \frac{T}{2}\delta(m-n)$.

5. $x(t) = \begin{cases} 1 & \text{if } 0 \leq |t| < \frac{\tau}{2} \\ 0 & \text{if } \frac{\tau}{2} < |t| \leq \frac{T}{2} \end{cases} \rightarrow x_k = \frac{\tau}{T}\frac{\sin(\pi k\tau/T)}{\pi k\tau/T}$. **Note:** Duty cycle$=\frac{\tau}{T}$.

COMPUTATION OF FOURIER SERIES USING INTEGRALS

Given: $x(t) = \begin{cases} +\pi/4 & \text{for } 0 < t < \pi \\ -\pi/4 & \text{for } \pi < t < 2\pi \end{cases}$ and periodic with period = T = 2π. **Goal** : Compute its Fourier series.

Hard Way:

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos(\tfrac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-\tfrac{\pi}{4})\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}(\tfrac{\pi}{4})\cos(nt)dt$$

$$= -\frac{1}{4n}\sin(nt)|_{-\pi}^{0} + \frac{1}{4n}\sin(nt)|_{0}^{\pi} = 0 - 0 + 0 - 0 = 0 \text{ since } \sin(n\pi) = 0.$$

Hard Way:

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin(\tfrac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-\tfrac{\pi}{4})\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}(\tfrac{\pi}{4})\sin(nt)dt$$

$$= \frac{1}{4n}\cos(nt)|_{-\pi}^{0} - \frac{1}{4n}\cos(nt)|_{0}^{\pi} = \frac{1}{4n}(1 - \cos(-\pi n)) - \frac{1}{4n}(\cos(\pi n) - 1)$$

$$= \frac{1}{n}\frac{1}{2}(1 - \cos(\pi n)) = \begin{cases} 1/n & \text{for } n \text{ odd;} \\ 0 & \text{for } n \text{ even} \end{cases} \text{ since } \cos(\pi n) = (-1)^n.$$

Note: This is awful! Isn't there any way to simplify this computation?

Def: An *even* function has $x(t) = x(-t) \Leftrightarrow$ *symmetric* about the $t = 0$ axis.
Ex: $\cos(\omega t), 1, t^2, t^4 \ldots$ Note $x(0)$ can be anything.
Def: An *odd* function has $x(t) = -x(-t) \Leftrightarrow$ *antisymmetric* about $t = 0$ axis.
Ex: $\sin(\omega t), t, t^3, t^5 \ldots$ Note $x(0) = 0$ if $x(0)$ defined.

So? $\int_{-any}^{+any} odd(t)dt = 0$ and $\int_{-any}^{+any} even(t)dt = 2\int_{0}^{+any} even(t)dt.$
Also: (even)(even)=even; (odd)(odd)=even; (even)(odd)=odd functions.
Here: Above $x(t)$ is an **odd** function (reflect it about both [not each] axes).
Then: Instead of computing four integrals, compute only one integral:

Try this again:

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos(\tfrac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(odd)(even)dt = 0.$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin(\tfrac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(odd)(odd)dt = \frac{2}{\pi}\int_{0}^{\pi}(even)dt$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi}(\tfrac{\pi}{4})\sin(nt)dt = \frac{1}{n}\frac{1}{2}(1 - \cos(\pi n)) = \text{the above result.}$$

Computation of Complex Exponential Fourier Series:

Still easier:

$$x_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)e^{-j\frac{2\pi}{T}nt}dt = \frac{1}{2\pi}\int_{-\pi}^{0}(-\tfrac{\pi}{4})e^{-jnt}dt + \frac{1}{2\pi}\int_{0}^{\pi}(\tfrac{\pi}{4})e^{-jnt}dt$$

$$= \frac{1}{8jn}e^{-jnt}|_{-\pi}^{0} - \frac{1}{8jn}e^{-jnt}|_{0}^{\pi} = \frac{1}{4jn}(1 - e^{-j\pi n}) = \frac{-j}{2n} = \frac{1}{2jn} \text{ if } n \text{ odd.}$$

Plug in:

$$x(t) = \frac{1}{2j}e^{+jt} + \frac{1}{6j}e^{+j3t} + \frac{1}{10j}e^{+j5t} + \frac{1}{14j}e^{+j7t} + \frac{1}{18j}e^{+j9t} + \ldots$$
$$- \frac{1}{2j}e^{-jt} - \frac{1}{6j}e^{-j3t} - \frac{1}{10j}e^{-j5t} - \frac{1}{14j}e^{-j7t} - \frac{1}{18j}e^{-j9t} - \ldots$$

$$\rightarrow x(t) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + \frac{1}{9}\sin(9t) + \ldots$$

LINE SPECTRA WITH SQUARES AND TIME DELAYS

Given: $x(t)$ has spectrum $3\delta[f]+2\delta[f+7]+2\delta[f-7]+4\delta[f+14]+4\delta[f-14]$
and: $y(t)=2+4\cos(14\pi t-\frac{\pi}{4})+12\cos^2(14\pi t)$.
Goal: Compute spectrum of $x(t+\frac{1}{56})+y(t)$. Here $\delta[f-7]$ =line at 7 Hz.

- $x(t)=3+4\cos(14\pi t)+8\cos(28\pi t)$ (note $\omega=2\pi f$; watch amplitudes).

$\rightarrow$ $x(t+\frac{1}{56})=3+4\cos(14\pi(t+\frac{1}{56}))+8\cos(28\pi(t+\frac{1}{56}))$
$\rightarrow$ $x(t+\frac{1}{56})=3+4\cos(14\pi t+\frac{\pi}{4})+8\cos(28\pi t+\frac{\pi}{2})$ (delay→phase).

- $y(t)=2+4\cos(14\pi t-\frac{\pi}{4})+12(\frac{1}{2}+\frac{1}{2}\cos(28\pi t))$ [Using $\cos^2(x)=$]

$\rightarrow$ $y(t)=8+4\cos(14\pi t-\frac{\pi}{4})+6\cos(28\pi t))$ [$\frac{1}{2}+\frac{1}{2}\cos(2x)$ here]

- Phasors of sum: **f=0:** 3+8=11. **f=7:** $4e^{j\pi/4}+4e^{-j\pi/4}=4\sqrt{2}$. **f=14:** $8e^{j\pi/2}+6=6+j8=10e^{j0.927}$. Line spectrum of sum:
- $x(t+\frac{1}{56})+y(t)=11+4\sqrt{2}\cos(14\pi t)+10\cos(28\pi t+0.927)$ has $11\delta[f]+2\sqrt{2}\delta[f+7]+2\sqrt{2}\delta[f-7]+5e^{-j0.927}\delta[f+14]+5e^{j0.927}\delta[f-14]$.

Parseval's Theorem: Power in Time **or** Frequency

Lemma: $MS(x+y)=MS(x)+MS(y)+\frac{1}{T}C(x,y)+\frac{1}{T}C(y,x)$, $MS(x)=\int|x(t)|^2dt$
Proof: $MS(x+y)=\frac{1}{T}\int|x+y|^2dt=\frac{1}{T}\int(x+y)(x+y)^*dt$, $C(x,y)=\int x(t)y(t)^*dt$
$=\frac{1}{T}\int xx^*+\frac{1}{T}\int yy^*+\frac{1}{T}\int xy^*+\frac{1}{T}\int yx^*$=above expression.

Def: $x(t)$ and $y(t)$ are *uncorrelated* (also known as) *orthogonal* if $C(x,y)=0$.
Corollary: $MS(x+y)=MS(x)+MS(y)$ if & only if $x(t)$ and $y(t)$ are *orthogonal*. Average power of sum=sum of average powers, for orthogonal signals.

Lemma: $\cos(i\omega_0 t)$ and $\cos(j\omega_0 t)$ are orthogonal unless $i=j$.
Lemma: $\cos(i\omega_0 t)$ and $\sin(j\omega_0 t)$ are orthogonal even if $i=j$.
Proof: See first Fourier series handout. **Note:** i and j must be **integers**.

Thm: **Parseval's Thm:** $\frac{1}{T}\int_0^T|x(t)|^2dt=a_0^2+\frac{1}{2}\sum(a_n^2+b_n^2)=\sum|x_n|^2$.
Proof: Average power of $a_n\cos(\frac{2\pi}{T}nt)=a_n^2/2$ unless $n=0$ (then it's a_0^2).
and: Average power of $b_n\sin(\frac{2\pi}{T}nt)=b_n^2/2$ (recall **rms** on a handout).
Now: Average power of $x(t)$=Average power of sum of its Fourier series
= Sum of average powers of terms of Fourier series since orthogonal.

EX: For above $x(t)$: $\frac{1}{T}\int_0^T|x(t)|^2dt=\frac{1}{2\pi}\int_0^{2\pi}|\pm\frac{\pi}{4}|^2dt=\pi^2/16$ (try it!)
and: $a_0^2+\frac{1}{2}\sum(a_n^2+b_n^2)=0+\frac{1}{2}(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)=\pi^2/16$ (try summing).
Complex $\sum|x_n|^2=2(|\frac{1}{2j}|^2+|\frac{1}{6j}|^2+|\frac{1}{10j}|^2+\ldots)=\frac{1}{2}(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)=\pi^2/16$.