EXAMPLES OF FIR DIGITAL FILTERS

**FIR:** FIR = Finite Impulse Response: \( h[n] \) has finite duration and support.

**Form:** \( y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \ldots + b_M x[n-M] \). Order = \( M \).

\( h[n] = b_n \) for \( 0 \leq n \leq M \); \( h[n] = 0 \) otherwise. Assume \( h[n] \) is real here.

Frequency response: \( H(e^{j\omega}) = b_0 + b_1 e^{-j\omega} + b_2 e^{-2j\omega} + \ldots + b_M e^{-jM\omega} \).

\( \cos(\omega_0 n) \rightarrow \text{FIR} \rightarrow H(e^{j\omega_0}) e^{j\omega_0 n} \), implying

\[ \cos(\omega_0 n) \rightarrow \text{FIR} \rightarrow |H(e^{j\omega_0})| \cos(\omega_0 n + \text{arg}[H(e^{j\omega_0})]) \]

Properties: \( H(e^{j\omega}) \) periodic in \( \omega \) with period \( 2\pi \).

of \( H(e^{j\omega}) \): Conjugate symmetry: \( H(e^{-j\omega}) = H^*(e^{j\omega}) \).

Implies: Gain is even function: \(|H(e^{j\omega})| = |H(e^{-j\omega})|\).

Phase is odd function: \( \text{arg}[H(e^{-j\omega})] = -\text{arg}[H(e^{j\omega})] \).

**Notch:** \( h[n] = \{1, -2 \cos(\omega_0), 1\} \rightarrow H(e^{j\omega}) = [2 \cos(\omega) - 2 \cos(\omega_0)] e^{-j\omega} \).

**Filter:** Implement using \( y[n] = x[n] - 2 \cos(\omega_0)x[n-1] + x[n-2] \).

**Does:** Eliminates component at a single frequency \( \omega_0 \) from input.

**Why?** Eliminate interference at a single frequency, e.g., 60 Hz “hum.”

**Comb** \( h[n] = \{1, 1, 1 \ldots 1\} \rightarrow H(e^{j\omega}) = \frac{e^{-j\omega(M+1)-1}}{e^{-j\omega-1}} = \frac{\sin((M+1)\omega)}{\sin(\omega)} e^{-j\omega M/2} \).

**Filter:** Implement using \( y[n] = x[n] + x[n-1] + \ldots + x[n-M] \).

**Does:** Eliminates components at frequencies \( \pm \frac{2\pi}{M+1}, \pm \frac{4\pi}{M+1} \ldots \pm \frac{M\pi}{M+1} \).

**Why?** Eliminate interference at harmonics of a single frequency \( 2\pi/(M+1) \).

e.g., Sawtooth wave interference with period \( (M+1) \) has these harmonics.

**Ideal** \( h[n] = \frac{\sin(Bn)}{\pi n} \rightarrow H(e^{j\omega}) = 1 \) for \( 0 < |\omega| < B \); \( = 0 \) for \( B < |\omega| < \pi \).

**LPF:** Implement using \( y[n] = \sum_{i=-\infty}^{\infty} \frac{\sin(Bi)}{\pi i} x[n - i] \); truncate the sum.

**Does:** Low-pass filter; eliminates frequencies above \( B \) (periodic; period \( 2\pi \)).

**Why?** Filter out high-frequency noise from a low-frequency signal (smooth).

APPLICATION TO ELECTROCARDIOGRAM (HEART) SIGNALS

**Given:** \( x(t) \) = EKG signal; periodic (you hope!) at 60 beats/minute = \( 1 \) beats/second.

**Goal:** Filter out 60 Hz interference from electrical outlet wires in lab.

**Signal** \( x(t) \) period = 1 second \( \rightarrow \) fundamental = 1 Hz \( \rightarrow \) harmonics = 2, 3, \ldots Hz.

**Huh?** \( x(t) = c_0 + c_1 \cos(2\pi t + \theta_1) + c_2 \cos(4\pi t + \theta_2) + c_3 \cos(6\pi t + \theta_3) \ldots \)

**DC** \underline{FUNDAMENTAL} \underline{1ST HARMONIC} \underline{2ND HARMONIC}

**Sample** at 1 kHz \( \rightarrow \Delta = 0.001 \) second \( \rightarrow t = 0.001n \rightarrow x[n] = x(t = 0.001n) \).

**Spectrum** of sampled signal \( x[n] = x(0.001n) \) is periodic in \( f \) with period \( 1 \) kHz.

**Notch** \( y[n] = x[n] - 2 \cos(2\pi \frac{60}{1000}) x[n-1] + x[n-2] \) eliminates 60 Hz “hum.”
ARMA DIFFERENCE EQUATIONS

MA: \( y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_q x[n-q] \) (Moving Average)

Huh? Present output = weighted average of \( q + 1 \) most recent inputs.

Note: Equivalent to \( y[n] = b[n] \ast x[n] \) where \( b[k] = b_k, 0 \leq k \leq q \).

AR: \( y[n] + a_1 y[n-1] + \ldots + a_p y[n-p] = x[n] \) (AutoRegression)

Huh? Present output = weighted sum of \( p \) most recent outputs.

Note: Compute \( y[n] \) recursively from its \( p \) most recent values.

\[
\begin{align*}
\text{ARMA: } & \sum_{i=0}^{p} a_i y[n-i] = \sum_{i=0}^{q} b_i x[n-i] \\
\text{AUTOREGRESSIVE} & \quad \text{MOVING AVERAGE}
\end{align*}
\]

Note: Difference equation \( \sim \) differential equation in continuous time:
\[
\left(\frac{d^p}{dt^p} + a_1 \frac{d^{p-1}}{dt^{p-1}} + \ldots + a_p\right) y(t) = \left(\frac{d^q}{dt^q} + b_1 \frac{d^{q-1}}{dt^{q-1}} + \ldots + b_q\right) x(t).
\]

Note: Coefficients \( a_i \) and \( b_i \) are not directly analogous here to \( a_i \) and \( b_i \) above.

Note: Linear time-invariant in both cases since all coefficients indpt. of time.

EXAMPLES OF IIR FILTERS

1. \( h[n] = ba^n u[n] \Leftrightarrow H(z) = b \frac{z^{-n}}{z-a} \Leftrightarrow y[n] = ay[n-1] = bx[n] \) \( [1^{st}\text{-order}] \).

2. \( h[n] = (c_1 p^n_1 + c_2 p^n_2) u[n] \Leftrightarrow H(z) = \frac{c_1 z^{-n_1}}{z-p_1} + \frac{c_2 z^{-n_2}}{z-p_2} = \frac{(c_1 + c_2) z^{-n} - (c_1 p_2 + c_2 p_1) z^{-n}}{z^n - (p_1 + p_2) z^{n-1} + (p_1 p_2)} \)
\( \Leftrightarrow y[n] = (p_1 + p_2)y[n-1] + (p_1 p_2)y[n-2] = (c_1 + c_2)x[n] - (c_1 p_2 + c_2 p_1)x[n-1] \)

2a. \( h[n] = 2 \cos(\omega_o n) u[n] = (e^{j \omega_o n} + e^{-j \omega_o n})u[n] \) \( [(2,1)^{nd\text{-order}} \text{ ARMA}] \)
\( \Leftrightarrow y[n] - 2 \cos(\omega_o) y[n-1] + y[n-2] = 2x[n] - 2 \cos(\omega_o) x[n-1]. \)

2b. \( h[n] = 6(\frac{2}{3})^n u[n] + 8(\frac{1}{3})^n u[n] \) can be implemented by: \([8(\frac{2}{3}) + 6(\frac{1}{3})] = 6 \)
\( y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = 14x[n] - 6x[n-1] \) \( [(2,1)^{nd\text{-order}} \text{ ARMA}] \).

SO WHY BOTHER TO USE IIR FILTERS?

Goal: Eliminate 60 Hz “hum” in a continuous-time signal sampled at 250 Hz.

FIR: Notch: \( y[n] = x[n] - 2 \cos(2\pi \frac{60}{250})x[n-1] + x[n-2] \). See below left.

IIR: ARMA: \( y[n] - 1.98 \cos(2\pi \frac{60}{250})y[n-1] + 0.98y[n-2] \) \( [(2,2)^{nd\text{-order}] \)
\( = x[n] - 2 \cos(2\pi \frac{60}{250})x[n-1] + x[n-2] \). See below right.

Point: The IIR filter has a much sharper frequency response than the FIR filter.