**THE DISCRETE FOURIER TRANSFORM (DFT)**

**NOTE:** See DFT: *Discrete Fourier Transform* for more details.

**Cont.** Let \( x(t) = x(t + T) \) be periodic with period \( T \) in **continuous** time.

**Time** Then \( x(t) \) can be expanded in the **continuous-time** Fourier series

\[
x(t) = X_0 + X_1 e^{j \frac{2\pi}{T} t} + X_2 e^{j \frac{4\pi}{T} t} + \ldots + X_{-1} e^{-j \frac{2\pi}{T} t} + X_{-2} e^{-j \frac{4\pi}{T} t} + \ldots
\]

Series where \( X_k = \frac{1}{T} \int_{t_0}^{t_0 + T} x(t) e^{-j \frac{2\pi}{T} k t} \, dt \) for integers \( k \) and any time \( t_0 \).

**Note:** Conjugate symmetry: \( x(t) \) real \( \iff X_{-k} = X_k^* \) for integers \( k \).

**Discrete** Let \( x[n] = x[n + N] \) be periodic with period \( N \) in **discrete** time.

**Time** Then \( x[n] \) can be expanded in the **discrete-time** Fourier series

\[
x[n] = X_0 + X_1 e^{j \frac{2\pi}{N} n} + X_2 e^{j \frac{4\pi}{N} n} + \ldots + X_{N-1} e^{j \frac{(N-1)2\pi}{N} n}
\]

Series where \( X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} \) for \( k = 0 \ldots N-1 \).

**Note:** Conjugate symmetry: \( x[n] \) real \( \iff X_{-k} = X_k^* \) for integers \( k \).

**EX:** \( x[n] = \{ \ldots 12, 6, 4, 6, 12, 6, 4, 6, 12, 6, 4, 6 \ldots \} \). Periodic; period \( N = 4 \).

**DFT:** \( X_0 = \frac{1}{4} (x[0] + (+1)x[1] + (+1)x[2] + (+1)x[3]) = \frac{1}{4} (12 + 6 + 4 + 6) = 7 \).

**DFT:** \( X_1 = \frac{1}{4} (x[0] + (-j)x[1] + (-1)x[2] + (+j)x[3]) = \frac{1}{4} (12 - 6j - 4 + 6j) = 2 \).

**DFT:** \( X_2 = \frac{1}{4} (x[0] + (-1)x[1] + (+1)x[2] + (-j)x[3]) = \frac{1}{4} (12 - 6 + 4 - 6) = 1 \).

**DFT:** \( X_3 = \frac{1}{4} (x[0] + (+j)x[1] + (-1)x[2] + (-j)x[3]) = \frac{1}{4} (12 + 6j - 4 - 6j) = 2 \).

**Note:** \( X_3 = X_{4-1} = X_1^* = 2^* = 2 \) (although both \( X_3 \) and \( X_1 \) are real here).

**Note:** \( x[n] \) is a real and even function \( \iff X_k \) is a real and even function.

Then: \( x[n] = 7 + 2e^{j \frac{\pi}{2} n} + 1e^{j\pi n} + 2e^{j \frac{3\pi}{2} n} \) (complex exponential form)

Or: \( x[n] = 7 + 4\cos(\frac{\pi}{2} n) + 1\cos(\pi n) \) (trigonometric form)

since: \( e^{j \frac{\pi}{2} n} = e^{-j \frac{\pi}{2} n} \) (try it) and \( e^{j\pi n} = \cos(\pi n) = (-1)^n \).

**Power:** Time domain: Average power \( \frac{1}{4} (12^2 + 6^2 + 4^2 + 6^2) = 58 \).

**Parseval:** Freq. domain: Average power \( (|1|^2 + |2|^2 + |1|^2 + |2|^2) = 58 \).

**So?** Compute average power in either time domain or frequency domain.

**So?** Consider \( x[n] \rightarrow \text{[LTI]} \rightarrow y[n] \) where input \( x[n] = \{ \ldots 12, 6, 4, 6 \ldots \} \)

and Linear Time-Invariant (LTI) system is \( y[n] = 3y[n-1] = 3x[n] + 3x[n-1] \).

Then: Frequency response function \( H(e^{j\omega}) = 3e^{j\omega-1} \) (Huh? stay tuned)

Then: \( H(e^{j0}) = 3 + \frac{1-1}{1-1} = 0 \);

\( H(e^{j\pi/2}) = 3 + \frac{j-1}{j-1} = 1.341e^{-j0.46} \)

and: \( H(e^{j\pi}) = 3 - \frac{1-1}{1-1} = \frac{3}{2} \);

\( H(e^{j3\pi/2}) = 3 - \frac{j-1}{j-1} = 1.341e^{j0.46} \)

Then: \( y[n] = (0)7 + 1.341e^{-j0.46}2e^{jn\pi/2} + \frac{3}{2}1e^{jn\pi} + 1.341e^{j0.46}2e^{jn3\pi/2} \)

and: \( y[n] = 7(0) + 4(1.341)\cos(\frac{\pi}{2} n - 0.46) + 1(\frac{3}{2})\cos(\pi n) \) which becomes

\( y[n] = 5.366\cos(\frac{\pi}{2} n - 0.46) + 1.5\cos(\pi n) \). Note DC term filtered out.
EXAMPLES OF DFT PROPERTIES

Given: \( x[n] \) is a discrete-time signal with period \( N: x[n] = x[n+N] \) for all \( n \).

DFT: \( x[n] = \sum_{k=0}^{N-1} X_k e^{i2\pi nk/N} \) where \( X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \)

1. \( X_0 = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \) DC value = mean value of periodic signal \( x[n] \).
2. Negative frequencies are second half of \( \{X_k\} \): Use \( X_{-k} = X_{N-k} \).
3. Matlab’s `fftshift` shifts DC to the center, from the left end of plot. This makes conjugate symmetry \( X_{-k} = X_{N-k} = X_k^* \) easier to see.
4. This is EECS 206 definition; \( \frac{1}{N} \) moved everywhere else. Matlab: `fft`.

EX #1: DFT\{1, 0, 0, 0, 0, 0, 0, 0\} = \( \frac{1}{8} \{1, 1, 1, 1, 1, 1, 1, 1\} \). Impulse in time.

EX #2: DFT\{0, 0, 0, 0, 0, 0, 0, 0\} = \( \frac{1}{8} \{1, -j, -1, j, 1, -j, -1, j\} \). Delayed \( \delta[n] \).

EX #3: DFT\{1, 1, 1, 1, 1, 1, 1, 1\} = \{1, 0, 0, 0, 0, 0, 0, 0\}. Constant in time.

EX #4: DFT\{1, 1, 1, 1, 1, 1, 1, 1\} = \( \frac{1}{8} \{9, -j, -1, j, 1, -j, -1, j\} \). DFT is linear.

Parseval: Average power = \( \|x[n]\|^2 = \frac{1}{2^2} = \frac{1}{8} (1^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2) = (\frac{1}{8})^2 (9 + -j^2 + -1^2 + j^2 + j^2 + -1^2 + j^2).

EX #5: DFT\{\cos(2\pi \frac{M}{N} n + \theta)\} = \frac{1}{2} e^{j\theta} \delta[k-M] + \frac{1}{2} e^{-j\theta} \delta[k-(N-M)] \).

Note: This only works for periodic discrete-time sinusoids: \( \omega_o = 2\pi \frac{M}{N} \).

EX: DFT\{24, 8, 12, 16\} = \{15, 3 + 2j, 3, 3 - 2j\} (1 period of \( x[n] \) and \( X_k \)).

\( x[n] = (15) e^{j0n} + (3 + 2j) e^{j(\pi/2)n} + (03) e^{j\pi n} + (3 - 2j) e^{j(3\pi/2)n} \).

1. DFT\{24, 16, 12, 8\} = \{15, 3 - 2j, 3, 3 + 2j\}. Reversal: \( x[-n] \rightarrow X_k^* \).

Huh? \( x[n] = \{\ldots 24, 8, 12, 16, 24, 8, 12, 16, 24, 8, 12, 16 \ldots \} \) → \( x[-n] = \{\ldots 24, 16, 12, 8, 24, 16, 12, 8, 24, 16, 12, 8 \ldots \} \).

2. DFT\{12, 16, 24, 8\} = \{15, -3 - 2j, 3, 2j - 3\}. Delay: \( x[n-D] \rightarrow x_k e^{-j2\pi kD/N} \).

Huh? \( x[n-2] = \{\ldots 12, 16, 24, 8, 12, 16, 24, 8, 12, 16, 24, 8 \ldots \} \).

3. DFT\{-24, -8, 12, -16\} = \{3, 3 - 2j, 15, 3 + 2j\}. \( x[n] e^{j2\pi kF/N} \rightarrow X_{k-F} \).

Huh? ”Modulate” signal means shift its spectrum by some frequency \( F \).

4. DFT\{24, 0, 8, 0, 12, 0, 16, 0\} = \{15, 3 + 2j, 3, 3 - 2j, 15, 3 + 2j, 3, 3 - 2j\}.

Huh? Interpolate with zeros → repeat and halve DFT of lower order.

5. DFT\{24, 8, 12, 16, 24, 8, 12, 16\} = \{15, 0, 3 + 2j, 0, 3, 0, 3 - 2j, 0\}.

Huh? Repeat in time → interpolate with zeros in frequency domain.