**GIVEN:** A periodic continuous-time signal \( x(t) \) such that:
1. \( x(t) \) is periodic and real: \( x(t) = x(t + T) \) for all \( t \);
2. \( x(t) \) is bandlimited: No frequencies above \( F \) Hz;
3. \( x(t) \) is sampled: Given samples \( x[n] = x(t = n\Delta) \).

**GOAL:** We can reconstruct \( x(t) \) from its samples \( x[n] = x(t = n\Delta) \).

**IF:** \( \Delta < 1/(2F) \iff \text{Sampling rate} > 2F \text{ samples/second} \).

**DERIVATION WITHOUT USING THE FOURIER TRANSFORM**

1. \( x(t) \) periodic with period \( T \rightarrow x(t) \) has the Fourier series expansion
   \[
   x(t) = X_0 + X_1 e^{j\frac{2\pi}{T}t} + X_2 e^{j\frac{2\pi}{T}t} + \ldots + X_N e^{j\frac{2\pi}{T}Nt} + X_1^* e^{-j\frac{2\pi}{T}t} + \ldots + X_N^* e^{-j\frac{2\pi}{T}Nt}
   \]
   where: \( X_k = \frac{1}{T} \int_0^T x(t) e^{-j\frac{2\pi}{T}kt} \, dt \) and \( \frac{N}{T} < F < \frac{N+1}{T} \). Say \( F = \frac{N+1/2}{T} \).

   **Note:** We will not need to use the formula for \( X_k \)! No integrals here!

2. Hence \( x(t) \) is specified by \( 2N+1 \) complex numbers \( \{X_{-N} \ldots X_0 \ldots X_N\} \).
3. Sample \( x(t) \) at \( t = n\Delta \) so there are \( 2N+1 \) samples per period \( T \).
   \[ \rightarrow (2N+1)\Delta = T. \] This and \( (N + \frac{1}{2}) = FT \rightarrow \Delta = \frac{1}{2F}. \]
4. Then setting \( t = n\Delta = \frac{n}{2F}, n = 0 \ldots 2N \) in the Fourier series
gives \( 2N+1 \) linear equations in \( 2N+1 \) unknowns \( \{X_k\} \):
5. \[ x(n\Delta) = \sum_{k=-N}^{N} X_k e^{\frac{j2\pi nk}{2N+1}}, n = 0 \ldots 2N. \] Sum over different period:
   \[ \rightarrow x(n\Delta) = \sum_{k=0}^{2N} X_k e^{\frac{j2\pi nk}{2N+1}}, n = 0 \ldots 2N \text{ (}(2N+1)-\text{point DFT}). \]
6. We can solve this linear system for the \( \{X_k\} \) and insert these \( \{X_k\} \) into the Fourier series in \#1 above to get \( x(t) \) for ALL values of \( t \).

**BANDLIMITED SIGNAL INTERPOLATION FORMULA**

7. Or, we can note that the solution to this linear system is:
   \[
   X_k = \frac{1}{2N+1} \sum_{n=0}^{2N} x(n\Delta) e^{-\frac{j2\pi nk}{2N+1}}, k = -N \ldots N \text{ (}(2N+1)-\text{point DFT})
   \]
8. Inserting this into the Fourier series and using \( \frac{t}{T} - \frac{n}{2N+1} = \frac{t-n\Delta}{T} \) gives
   \[
   x(t) = \sum_{k=-N}^{N} \frac{1}{2N+1} \sum_{n=0}^{2N} x(n\Delta) e^{-\frac{j2\pi nk}{2N+1}} e^{j\frac{2\pi}{T}kt} = \sum_{n=0}^{2N} x(n\Delta) s(t - n\Delta)
   \]
   where \( s(t) = \frac{1}{2N+1} \sum_{k=-N}^{N} e^{j2\pi kt/T} = \frac{\sin[(2N+1)\pi t/T]}{(2N+1)\sin(\pi t/T)} \) (text p.145).

9. **NOTE:** This holds for any value of \( T \), e.g., \( T=1 \text{ century}! \)
10. Shannon proved this for aperiodic signals (think of this as \( T \rightarrow \infty \)).

   **Note:** As \( T \rightarrow \infty, s(t) \rightarrow \frac{\sin(\pi t/\Delta)}{\pi t/\Delta} = \text{sinc}(t/\Delta) = p(t) \) in the lecture notes
   since: \( \frac{2N+1}{T} = \frac{1}{\Delta} \) and \( \sin(\pi t/T) \approx (\pi t/T) \) as \( \frac{\pi t}{T} \rightarrow 0 \iff (T \rightarrow \infty) \).

**EX:** \( x(t) \) has period=\( T=10 \text{ sec} \); bandlimit=\( F=100 \text{ Hz} \). Then \( \Delta = \frac{1}{200}. \)

   **since:** Fourier series of \( x(t) \) has 2001 terms \( \rightarrow 2001 \) samples per period=\( T \).
DIGITAL SIGNAL PROCESSING: COMPLETE SYSTEM

\[ x(t) = \text{Continuous-time (analog) signal.} \]

**EX:** Audio (from microphone) signal.

**Goal:** Digitally filter this signal \( x(t) \).

\[ \tilde{x}(t) = \text{Lowpass-filtered version of } x(t). \]

**How?** Analog filter; use EECS 215 ideas.

**Why?** Remove frequencies \( > \frac{1}{2}(\text{SAMPLING FREQUENCY}) \)
\( \rightarrow \) ensures there will be no aliasing.

\[ x[n] = \text{Discrete-time (sampled) signal.} \]

**How?** \( x[n] = \tilde{x}(t = n\Delta); \Delta = \text{SAMPLING INTERVAL}. \)

**Why?** We can now use EECS 216 ideas.

**Note:** Can recover \( \tilde{x}(t) \) from \( x[n] \) exactly, due to the anti-alias filter.

\[ \hat{x}[n] = \text{Quantized version of } x[n]. \]

**How?** Round \( x[n] \) to nearest of \( 2^B \) levels.
\( B = \# \text{bits used to represent numbers.} \)

**Why?** To send/store bits, not numbers.

**Note:** Can’t recover \( x[n] \) from \( \hat{x}[n] \), but the error is usually negligible.

\[ y[n] = \text{Filtered version of input } \hat{x}[n]. \]

**How?** \( y[n] = h[n] \ast \hat{x}[n] = \sum_i h[i]\hat{x}[n - i]. \)

**Why?** Lowpass filter—remove some noise.
Notch filter—remove 60 Hz “hum.”

\[ y(t) = \text{Interpolated } y[n] = \text{analog output.} \]

**How?** Use zero-order hold (constant interpolation)
OR: Linear interpolation (between samples \( y[n] \))
OR: Exact formula (Sampling handout).
ALIASING IN A COMPLETE DSP SYSTEM

Given: \( x(t) \rightarrow \text{SAMPLE AT 5 HZ} \rightarrow x[n] \rightarrow \text{IDEAL (SINC) INTERPOLATOR} \rightarrow \hat{x}(t) \)

where: \( x(t) = \cos(2\pi t) + 2 \cos(8\pi t) \) (1 Hz, 4 Hz). \textbf{GOAL:} Compute \( \hat{x}(t) \).

\textbf{Ideal Interpolator:} \( \hat{x}(t) = \sum x[n]p(t - nT_s) \) where \( p(t) = \text{sinc}(t/T_s) \).
\textbf{sinc:} \( \text{sinc}(t) = (\sin(\pi t))/((\pi t)) = \text{decaying sinusoid as } t \to \pm \infty. \)

\textbf{Nyquist:} Sampling rate=5 Hz<2(max. frequency of \( x(t) \)=2(4Hz)) \to \text{aliasing.}
\textbf{Interval:} Sampling rate=5 Hz \to T_s=\text{Sampling interval}=1/(5 Hz)=\frac{1}{5}\text{ second.}
\textbf{Sample:} \( t = nT_s = n(\frac{1}{5}) \to x[n] = x(n) = \cos(0.4\pi n) + 2 \cos(1.6\pi n) \).

\textbf{Alias:} \( 2 \cos(1.6\pi n) = 2 \cos(0.4\pi n) \to x[n] = 3 \cos(0.4\pi n) \). Note tripled!
\textbf{Ideal:} \( n = \frac{t}{T_s} = 5t \to \hat{x}(t) = x[n = 5t] = 3 \cos(2\pi t) \) (1 Hz, but \textbf{tripled}).
\textbf{Note:} Original 4 Hz \to \text{aliased} 1 Hz (folded across folding freq. = \frac{5}{2} = 2.5 Hz).

\textbf{Now:} Change \( x(t) \) to \( x(t) = \cos(2\pi t) - \cos(8\pi t) \) (1 Hz, 4 Hz).
\textbf{Alias:} \( x[n] = \cos(0.4\pi n) - \cos(0.4\pi n) = 0! \text{ 1 Hz } \text{eliminated!} \)
\textbf{Aliasing:} adds false signals, interferes with actual signal!

\textbf{Now:} Insert ideal antialias filter: Lowpass; pass<2.5 Hz, reject>2.5 Hz.

Given: \( x(t) \rightarrow \text{ANTI-ALIAS} \rightarrow \text{SAMPLE AT 5 HZ} \rightarrow x[n] \rightarrow \text{IDEAL (SINC) INTERPOLATOR} \rightarrow \hat{x}(t) \)

\textbf{Now:} Antialias filter eliminates original 2 \( \cos(8\pi t) \) (4 Hz) component.
\textbf{Get:} \( x[n] = \cos(0.4\pi n) \) and \( \hat{x}(t) = \cos(2\pi t) \) (1 Hz).
\textbf{Note:} Aliased (false) 1 Hz eliminated. Original 1 Hz \textbf{unaffected}, at least.

\textbf{Alias:} Use \( A \cos((\pi + \omega_o)n + \theta) = A \cos((\omega_o - \pi)n + \theta) = A \cos((\pi - \omega_o)n - \theta) \)
\textbf{Since:} \( \cos(t) \) is an even function, and also periodic with period 2\( \pi \).
\textbf{EX:} \( 3 \cos(1.7\pi n + \frac{\pi}{6}) = 3 \cos(0.3\pi n - \frac{\pi}{6}) \).
\( \sin(1.8\pi n) = -\sin(0.2\pi n) \).

- Use to reduce all discrete-time signals \textbf{resulting from sampling.}
- For \textbf{non-sinusoidal signals:} Apply to Fourier series harmonics.

\textbf{MSD:} \( \text{MSD}(x, \hat{x}) = \frac{1}{T} \int_0^T (x(t) - \hat{x}(t))^2 dt = \text{Mean Square Error.} \)

\textbf{How?} Use Parseval’s theorem to add average power in each harmonic:
\textbf{Note:} Average power of \( A \cos(\omega_o n + \theta) \) is \( A^2/2 \).
\textbf{IF:} \( 1 \) \( \omega_0 = 2\pi(T^{\text{RATIONAL NUMBER}}) \) \to \text{periodic; } 2 \) \( \omega_0 \neq 0, \pi. \)
The figure shows the spectra of the functions:

- $\cos(\pi n/10)$
- $\cos(19\pi n/10)$
- $\cos(10\pi n/10)$
- $\cos(20\pi n/10)$

The spectra are plotted for $t=n/10$ and $t=n/5$, with frequencies in the range $[1 \text{ Hz}, 3 \text{ Hz}]$. The plots demonstrate the discrete nature of the sampled frequencies and the periodicity of the cosine functions.