FORWARD OR DIRECT Z-TRANSFORM

DEF: $\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$ (compare to $\mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$).

Huh? *Finite*-duration signal $x[n]$ $\rightarrow$polynomial $X(z)$ with coefficients$=x[n]$.

Or: *Infinite*-duration $x[n]$=coefficients of Laurent (power) series $X(z)$.

Finite length signal $x[n] = \{\underline{3}, 1, 4, 2, 5\}$. Note $x[0] = 3$ and finite duration=5$\rightarrow$
$X(z) = 3 + 1z^{-1} + 4z^{-2} + 2z^{-3} + 5z^{-4} = (3z^4 + z^3 + 4z^2 + 2z + 5)/z^4$.
$\mathcal{Z}\{\delta[n-D]\} = z^{-D}$ for any integer delay $D \geq 0$. $\mathcal{Z}\{\delta[n]\} = 1$.

Exponential $\mathcal{Z}\{a^n u[n]\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n = 1/(1-az^{-1}) = z/(z-a)$.
EX: $\mathcal{Z}\{(\frac{1}{2})^n u[n]\} = \frac{1}{1-\frac{1}{2}z^{-1}}$. EX: $\mathcal{Z}\{(-\frac{1}{3})^n u[n]\} = \frac{1}{1+\frac{1}{3}z^{-1}}$.

Sinusoids $\mathcal{Z}\{\cos(\omega_0 n)u[n]\} = \frac{1}{2}\mathcal{Z}\{e^{j\omega_0 n}u[n]\} + \frac{1}{2}\mathcal{Z}\{e^{-j\omega_0 n}u[n]\} =$ (since linear)
$\frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}} = \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}} = \frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$.

Linear: $\mathcal{Z}\{\{\underline{3}, 1, 4\} + u[n]\} = \frac{3z^2+1z+4}{z^2} + \frac{z}{z-1} = \frac{4z^3-2z^2+3z-4}{z^3-z^2} = \frac{\text{RATIONAL}}{\text{FUNCTION}}$.

Delay: If $\mathcal{Z}\{x[n]\} = X(z)$ and $D \geq 0$, then $\mathcal{Z}\{x[n-D]\} = z^{-D}X(z)$.

EX: $\mathcal{Z}\{u[n] - u[n-2]\} = \frac{z}{z-1} - z^{-2}\frac{z}{z-1} = \frac{z}{z^2}\frac{z^2-1}{z-1} = 1 + z^{-1}$.

Note: Note that $u[n] - u[n-2] = \delta[n] + \delta[n-1]$, so these are consistent.

Also: $\mathcal{Z}\{\frac{1}{n}u[n-1]\} = \sum_{n=1}^{\infty}\frac{z^{-n}}{n} = -\log(1-z^{-1})$ if you recognize this.

Note: Convolution$\leftrightarrow$polynomial multiplication: $\mathcal{Z}\{x[n] * y[n]\} = X(z)Y(z)$.

Eigenfuncs of LTI $z^n \rightarrow \boxed{h[n]} \rightarrow z^n H(z)$: z^n in$\rightarrow$ scaled z^n out. $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$.
$h[n]*z^n = \sum h[i]z^{n-i} = z^n \sum h[i]z^{-i} = z^n H(z)$. $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$.
Note z^n plays the same role that e^{st} plays in continuous time.

Note: $H(z)$=**transfer function**. $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$=**frequency response**.

EX: Compute step response (to $u[n]$) of LTI system with $h[n] = \{\underline{2}, -3, 1\}$.

Soln: We need to compute $y[n] = \{\underline{2}, -3, 1\} * u[n]$. **Do this two ways:**

#1: $y[n] = (2\delta[n] - 3\delta[n-1] + 1\delta[n-2]) * u[n]$. Using $u[n] * \delta[n] = u[n]$,

#1: $y[n] = 2u[n] - 3u[n-1] + u[n-2] = \{\underline{2}, -1\}$ (try it!) has duration=2.

#2: $Y(z) = H(z)U(z) = (2-3z^{-1}+z^{-2})\frac{1}{1-z^{-1}} = 2-z^{-1} \rightarrow y[n] = \{\underline{2}, -1\}$.

Note: This system has wiped out the step input! This seldom happens.

APPLICATIONS OF THE z-TRANSFORM

Given: $x[n] = 3^n u[n] \rightarrow \boxed{y[n] - 2y[n-1] = x[n-1] - x[n-2]} \rightarrow y[n]$

Goal: Compute the response $y[n]$ of the system to this particular input.

$\mathcal{Z}$: $Y(z) - 2z^{-1}Y(z) = z^{-1}X(z) - z^{-2}X(z)$ and $X(z) = \frac{z}{z-3}$ here

$\rightarrow Y(z) = \frac{z^{-1}-z^{-2}}{1-2z^{-1}}\frac{z}{z-3} = \frac{z-1}{(z-2)(z-3)} = \frac{2}{z-3} - \frac{1}{z-2} \quad [2 = \frac{3-1}{3-2}; -1 = \frac{2-1}{2-3}]$

$\rightarrow y[n] = [2(3)^{n-1} - (2)^{n-1}]u[n-1] = \substack{\text{FORCED}\\\text{RESPONSE}}(\substack{\text{like}\\x[n]}) + \substack{\text{NATURAL}\\\text{RESPONSE}}(\substack{\text{like}\\h[n]})$.

Given: $x[n] = (\frac{1}{2})^n u[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = \{\underline{0}, 0, 1\} = \delta[n-2]$

Goal: Compute the response of this system to $2\cos(\frac{\pi}{3}n)$.

H(z): $\substack{\text{TRANSFER}\\\text{FUNCTION}} = H(z) = \mathcal{Z}\{y[n]\}/\mathcal{Z}\{x[n]\} = z^{-2}/[z/(z-\frac{1}{2})] = (z-\frac{1}{2})/z^3$.

h[n]: $\substack{\text{IMPULSE}\\\text{RESPONSE}} = h[n] = \mathcal{Z}^{-1}\{(z-\frac{1}{2})/z^3\} = \mathcal{Z}^{-1}\{z^{-2}-\frac{1}{2}z^{-3}\} = \{\underline{0}, 0, 1, -\frac{1}{2}\}$.

H(w): $\substack{\text{FREQUENCY}\\\text{RESPONSE}} = H(\omega) = H(z)|_{z=e^{j\omega}} = (e^{j\omega} - \frac{1}{2})/e^{j3\omega}$.

$\omega = \frac{\pi}{3}$: $H(\frac{\pi}{3}) = [e^{j\pi/3} - \frac{1}{2}]/e^{j3\pi/3} = (\frac{1}{2} + j\frac{\sqrt{3}}{2} - \frac{1}{2})/(-1) = -j\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}e^{-j\pi/2}$.

Sol'n: $2\cos(\frac{\pi}{3}n) \rightarrow \boxed{\textbf{LTI}} \rightarrow \sqrt{3}\cos(\frac{\pi}{3}n - \frac{\pi}{2}) = \sqrt{3}\sin(\frac{\pi}{3}n)$.

Given: $x[n] \rightarrow \boxed{y[n] = x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]} \rightarrow y[n]$

Huh? $x[n]$=cell phone signal. $y[n]$=multipath due to buildings.

Goal: Compute the **inverse filter** that recovers $x[n]$ from $y[n]$:

Huh? $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \rightarrow \boxed{g[n]} \rightarrow x[n]$. That is, $g[n]$ *undoes* $h[n]$.

Idea: Systems in cascade (series) $\Leftrightarrow h[n] * g[n] = \delta[n] \Leftrightarrow H(z)G(z) = 1$.

Here: $h[n] = \{\underline{1}, -\frac{3}{4}, \frac{1}{8}\} \rightarrow H(z) = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = (z^2 - \frac{3}{4}z + \frac{1}{8})/z^2$.

$\rightarrow G(z) = 1/H(z) = z^2/[z^2 - \frac{3}{4}z + \frac{1}{8}] = z^2/[(z-\frac{1}{2})(z-\frac{1}{4})]$.

$\mathcal{Z}^{-1}$: $\frac{G(z)}{z} = \frac{z}{(z-1/2)(z-1/4)} = \frac{2}{z-1/2} - \frac{1}{z-1/4} \rightarrow G(z) = 2\frac{z}{z-1/2} - 1\frac{z}{z-1/4}$.

using: residues $2 = (1/2)/[(1/2) - (1/4)]$ and $-1 = (1/4)/[(1/4) - (1/2)]$.

g[n]: $g[n] = 2(\frac{1}{2})^n u[n] - (\frac{1}{4})^n u[n]$=inverse filter for original system.

Note: Stable since zeros of H(z)=poles of G(z) are inside unit circle.

Note: $g[0] \neq 0$ since both $\substack{\text{numerator \&}\\\text{denominator}}$ of $G(z)$ have the same degrees.