

Z-Transforms, Their Inverses

Transfer or System Functions

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I. ABSTRACT

The purpose of this document is to introduce EECS 206 students to the z-transform and what it's for. It also introduces transfer (system) functions and shows how to use them to relate system descriptions.

A. Table of contents by sections:

1. Abstract (you're reading this now)
2. Forward Z-Transforms: How do I compute z-transforms?
3. Inverse Z-Transforms: How do I "undo" a z-transform?
4. Transfer (System) Functions: What are they for?
5. Poles and Zeros: Transient and Frequency Responses
6. The Atlanta Airport: (Has he completely lost his mind?)
7. Half-Dozen Examples: This stuff is actually very useful!

Why should you care about z-transforms and transfer functions? Because of problems like this:

Given that the response of an LTI system to input $(\frac{1}{2})^n u[n]$ is output $\{0, 0, 1\} = \delta[n - 2]$,

Compute the response of the system to input $2 \cos(\frac{\pi}{3}n)$. That is:

If $x[n] = (\frac{1}{2})^n u[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = \{0, 0, 1\}$, then $x[n] = 2 \cos(\frac{\pi}{3}n) \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = ?$

By the end of this document, we will solve this very problem, and quite easily. How? Read on...

II. FORWARD Z-TRANSFORMS

We consider the cases of finite-duration and infinite-duration signals separately.

A. Finite-Duration Signals

Let $x[n]$ be a discrete-time signal which is:

- Causal ($x[n] = 0$ for $n < 0$)
- Finite Duration ($x[n] = 0$ for *all* $n > N$ for some N)

The z -transform of $x[n]$ is the polynomial (in z^{-1})

$$\mathcal{Z}\{x[n]\} = X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots + x[N]z^{-N} = (x[0]z^N + x[1]z^{N-1} + \dots + x[N])/z^N \quad (1)$$

Why do z -transforms use z^{-1} instead of z ? “It’s always something”—Gilda Radner on *Saturday Night Live*.
But there is a good reason for this—see below under “Poles and Zeros.”

Three important properties of the z -transform that follow from its definition:

- **Linearity:** $\mathcal{Z}\{ax[n] + by[n]\} = aX(z) + bY(z)$ where $X(z) = \mathcal{Z}\{x[n]\}$ and $Y(z) = \mathcal{Z}\{y[n]\}$
 - Use this to break up computation of z -transform (and their inverses)
- **Delays:** $\mathcal{Z}\{x[n - D]\} = z^{-D}\mathcal{Z}\{x[n]\} = z^{-D}X(z)$
 - Delaying a signal by $D > 0$ multiplies its z -transform by z^{-D}
 - But remember that the z -transform is only defined for causal signals
- **Convolution:** $\mathcal{Z}\{x[n] * y[n]\} = X(z)Y(z)$ where $X(z) = \mathcal{Z}\{x[n]\}$ and $Y(z) = \mathcal{Z}\{y[n]\}$
 - This follows since convolution is polynomial multiplication
 - Just think—no more convolutions to compute! You should like \mathcal{Z} already

Two examples that pretty much tell the whole story for finite-duration signals:

$$\begin{aligned} \mathcal{Z}\{\underline{3}, 1, 4, 2, 5\} &= 3 + 1z^{-1} + 4z^{-2} + 2z^{-3} + 5z^{-4} = 3z^4 + 1z^3 + 4z^2 + 2z + 5/z^5 \\ \mathcal{Z}\{2\delta[n] + 7\delta[n - 2] + 3\delta[n - 5]\} &= 2 + 7z^{-2} + 3z^{-5} = (2z^5 + 7z^3 + 3)/z^5 \end{aligned} \quad (2)$$

Recall that the underline in the first example shows the location of time $n = 0$. The second example can be performed either directly from the definition, or by using the linearity and delay properties on $\mathcal{Z}\{\delta[n]\} = 1$.

B. Infinite-Duration Signals

Now let $x[n]$ be a causal signal of infinite duration (i.e., $x[n]$ can be nonzero for arbitrarily large time n). The z -transform of $x[n]$ is now the power series

$$\mathcal{Z}\{x[n]\} = X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots = \sum_{n=0}^{\infty} x[n]z^{-n} \quad (3)$$

If you know what a Laplace transform is, this should look like a discrete-time version of it, as indeed it is. If you don’t know what a Laplace transform is, you aren’t missing anything that will help you.

The above three properties still hold, but computing $X(z)$ now requires use of:

$$1 + r + r^2 + r^3 + \dots = 1/(1 - r) \quad \mathbf{IF} \quad |r| < 1 \quad (4)$$

You will use this equation extensively in this document! Starting right now:

$$\mathcal{Z}\{a^n u[n]\} = 1 + az^{-1} + a^2 z^{-2} + \dots = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a/z)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad (5)$$

This formula is only true for $|a/z| < 1 \rightarrow |z| > a$. This is called the *region of convergence* (ROC) of the z-transform. In EECS 206 this is fine print that you can ignore. In EECS 451 things will be *very* different!

Recall $u[n]$ is the unit step function: $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ for $n < 0$. Multiplication by $u[n]$ simply takes the causal part of a^n . In fact, note that

$$\mathcal{Z}\{u[n]\} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \quad (6)$$

C. Examples of Infinite-Duration Signals

A simple example, just plugging into this formula:

$$\mathcal{Z}\{3^n u[n]\} = \frac{1}{1 - 3z^{-1}} = \frac{z}{z - 3} \quad (7)$$

A more interesting example that uses linearity and delay properties and illustrates an important point:

$$\mathcal{Z}\{u[n] - u[n - 2]\} = \frac{z}{z - 1} \frac{z}{z} - z^{-2} \frac{z}{z - 1} = \frac{z^2 - 1}{z(z - 1)} = \frac{z + 1}{z} = 1 + z^{-1} \quad (8)$$

But that looks like the z-transform of $\{\underline{1}, 1\}$! Did we make a mistake? No, because

$$u[n] - u[n - 2] = \{\underline{1}, 1\} = \delta[n] + \delta[n - 1] \quad (9)$$

This is one of many things about EECS 206 that can drive you crazy—getting the right answer and not knowing it. This happens because there is more than one way of representing a signal. While this flexibility is very useful, it can be aggravating. Try plotting your result using Matlab to see if it matches the “correct” answer, which may just be your answer in a different form.

A complex (snicker) example, using linearity:

$$\mathcal{Z}\{\cos(\omega n)u[n]\} = \mathcal{Z}\left\{\frac{1}{2}[e^{j\omega n} + e^{-j\omega n}]u[n]\right\} = \frac{1}{2}\mathcal{Z}\{e^{j\omega n}u[n]\} + \frac{1}{2}\mathcal{Z}\{e^{-j\omega n}u[n]\} = \frac{1}{2}\frac{z}{z - e^{j\omega}} + \frac{1}{2}\frac{z}{z - e^{-j\omega}} \quad (10)$$

This is algebraically correct, but we can put the two terms over a common denominator:

$$X(z) = \frac{1}{2}\frac{z}{z - e^{j\omega}}\frac{z - e^{-j\omega}}{z - e^{-j\omega}} + \frac{1}{2}\frac{z}{z - e^{-j\omega}}\frac{z - e^{j\omega}}{z - e^{j\omega}} = \frac{1}{2}\frac{2z^2 - ze^{-j\omega} - ze^{j\omega}}{z^2 - ze^{-j\omega} - ze^{j\omega} + 1} = \frac{z^2 - z\cos(\omega)}{z^2 - 2z\cos(\omega) + 1} \quad (11)$$

An example of plugging into this formula:

$$\mathcal{Z}\{\cos((\pi/3)n)u[n]\} = \frac{z^2 - z \cos(\pi/3)}{z^2 - 2z \cos(\pi/3) + 1} = \frac{z^2 - 0.5z}{z^2 - z + 1} \quad (12)$$

since $\cos(\pi/3) = (1/2)$. Note the ROC is $|z| > |e^{j\omega}| = 1$ for any ω .

D. Finite-Duration-Plus-Infinite Duration

The significance of linearity of the z-transform is shown in these two examples:

$$\mathcal{Z}\{\{\underline{3}, 1, 4\} + u[n]\} = \frac{3z^2 + 1z + 4}{z^2} \frac{z - 1}{z - 1} + \frac{z}{z - 1} \frac{z^2}{z^2} = \frac{4z^3 - 2z^2 + 3z - 4}{z^3 - z^2} \quad (13)$$

Simple enough, but try this:

$$\mathcal{Z}\{2 \cos((\pi/3)n)u[n] - \{\underline{2}, 1\}\} = 2\mathcal{Z}\{\cos((\pi/3)n)u[n]\} - \mathcal{Z}\{\underline{2}, 1\} = \frac{2z^2 - z}{z^2 - z + 1} - (2 + z^{-1}) \quad (14)$$

Again, this is algebraically correct, but we can put both terms over a common denominator:

$$X(z) = \frac{2z^2 - z}{z^2 - z + 1} - \frac{2z + 1}{z} = \frac{2z^2 - z}{z^2 - z + 1} \frac{z}{z} - \frac{2z + 1}{z} \frac{z^2 - z + 1}{z^2 - z + 1} = -\frac{z + 1}{z^3 - z^2 + z} \quad (15)$$

Note that the degree of the numerator is less than the degree of the denominator. This is called a *strictly proper* function, and it tells us that $x[0] = 0$. Why? For any causal function $x[n]$, we have

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} (x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots) = x[0] \quad (16)$$

This is called the *Initial Value Theorem* for the z-transform. It tells us:

- If $\text{degree}[\text{numerator}] < \text{degree}[\text{denominator}]$ (strictly proper $X(z)$), then $x[0] = 0$
- If $\text{degree}[\text{numerator}] = \text{degree}[\text{denominator}]$ (proper $X(z)$), then $x[0] \neq 0$
- If $\text{degree}[\text{numerator}] > \text{degree}[\text{denominator}]$ (improper $X(z)$), then $X(z)$ is wrong!

The above $X(z)$ has $\text{degree}[\text{numerator}] = \text{degree}[\text{denominator}] - 2$. This implies $x[0] = x[1] = 0$. Indeed, note that $x[n]$ was constructed so that this would be the case.

Why this obsession with writing $X(z)$ as the ratio of two polynomials? You'll find out in the next section. This is called a *rational function*, just as the ratio of two integers is called a rational number.

E. Application of Z-Transform

Already we can use z-transforms to make our life easier. Consider the following problem:

Given: An LTI system is described by the *difference equation* $y[n] - 2y[n-1] = x[n-1] - x[n-2]$.

Goal: Compute the output $y[n]$ if the input $x[n] = 3^n u[n]$. That is,

$$x[n] = 3^n u[n] \rightarrow \boxed{y[n] - 2y[n-1] = x[n-1] - x[n-2]} \rightarrow y[n] = ? \quad (17)$$

Solution: Take the z -transforms of the difference equation and of the input. This yields

$$Y(z) - 2z^{-1}Y(z) = z^{-1}X(z) - z^{-2}X(z); \quad X(z) = \frac{z}{z-3} \quad (18)$$

Inserting the second equation into the first gives

$$Y(z) = \frac{z^{-1} - z^{-2}}{1 - 2z^{-1}} \frac{z}{z-3} = \frac{z-1}{(z-2)(z-3)} = \frac{2}{z-3} - \frac{1}{z-2} \quad (19)$$

and just looking at the last expression, we recognize that

$$y[n] = 2(3)^{n-1}u[n-1] - (2)^{n-1}u[n-1] = \text{FORCED RESPONSE} \left(\begin{array}{l} \text{like} \\ x[n] \end{array} \right) + \text{NATURAL RESPONSE} \left(\begin{array}{l} \text{like} \\ h[n] \end{array} \right) \quad (20)$$

Note that $y[n]$ has a term that looks like the input (forced response) and a term that looks like the impulse response (natural response). This almost always happens.

How did I get that last expression from the next-to-last one? Read on...

III. INVERSE Z-TRANSFORMS

As long as $x[n]$ is constrained to be causal ($x[n] = 0$ for $n < 0$), then the z -transform is invertible: There is only one $x[n]$ having a given z -transform $X(z)$. Inversion of the z -transform (getting $x[n]$ back from $X(z)$) is accomplished by recognition: What $x[n]$ would produce that $X(z)$? Linearity of the z -transform allows us to break up $X(z)$ into parts, each of which can be inverted separately. But usually we will have to use *partial fractions* to accomplish this (see below).

A. Inversion by Recognition

Sometimes z -transform inversion is easy:

$$\mathcal{Z}^{-1}\{2z^4 + 7z^3 + z^2 + 8z + 3\}/z^4 = \mathcal{Z}^{-1}\{2 + 7z^{-1} + z^{-2} + 8z^{-3} + 3z^{-4}\} = \{2, 7, 1, 8, 3\} \quad (21)$$

As before, this can be performed either directly from the definition, or by using the linearity and delay properties on $\mathcal{Z}^{-1}\{1\} = \delta[n]$. An example with different types of terms to recognize:

$$\mathcal{Z}^{-1}\left\{\frac{4z+5}{z^2} + \frac{6}{1-3z^{-1}}\right\} = \mathcal{Z}^{-1}\{4z^{-1} + 5z^{-2}\} + 6\mathcal{Z}^{-1}\left\{\frac{1}{1-3z^{-1}}\right\} = \{0, 4, 5\} + 6(3)^n u[n] \quad (22)$$

Don't forget the step functions $u[n]$ —your inverse z -transform must be zero for $n < 0$. Otherwise it's wrong half the time, and you should only get half credit for it on an exam!

$$\mathcal{Z}^{-1}\left\{\frac{7z}{z-1} + \frac{8z}{z-2}\right\} = 7\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} + 8\mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\} = 7u[n] + 8(2)^n u[n] \quad (23)$$

This works if your $X(z)$ happens to be written as a sum of terms of the forms $\frac{bz}{z-a}$: Each term becomes $ba^n u[n]$. How do we make that happen? This should look familiar—perform a *partial fraction expansion* of $X(z)$. But there’s a twist involved—see below.

B. Review of Partial Fraction Expansions

Any *rational function* $X(z)$ that is *strictly proper* (both terms defined above) can be expanded as

$$X(z) = \frac{(z - z_1) \dots (z - z_M)}{(z - p_1) \dots (z - p_N)} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \dots + \frac{A_N}{z - p_N} \quad (24)$$

- “Strictly proper” here means $N > M$ in the rational function $X(z)$.
- The **poles** $\{p_n\}$ are the roots of the denominator polynomial=0.
- The **poles** $\{p_n\}$ are assumed to be distinct (no two are equal);
- Repeated roots of polynomials don’t occur in real life unless forced to.
- If $X(z)$ is real, both poles and residues occur in complex conjugate pairs:
- If A_n is the residue for p_n , then A_n^* is the residue for p_n^* .
- The **residues** $\{A_n\}$ are computed using the following formula:

$$A_n = X(z)(z - p_n)|_{z=p_n} = \frac{(p_n - z_1) \dots (p_n - z_n) \dots (p_n - z_M)}{(p_n - p_1) \dots (p_n - p_{n-1})(p_n - p_{n+1}) \dots (p_n - p_N)} \quad (25)$$

A simple example: Compute the partial fraction expansion of $\frac{z-3}{z^2-3z+2}$.

1. Compute the **poles**: $z^2 - 3z + 2 = (z - 1)(z - 2) = 0 \rightarrow p_1 = 1$ and $p_2 = 2$
2. Compute the **residues**: $A_1 = \frac{1-3}{1-2} = 2$ and $A_2 = \frac{2-3}{2-1} = -1$
3. Plug in: $\frac{z-3}{z^2-3z+2} = \frac{2}{z-1} - \frac{1}{z-2}$ (confirm by cross-multiplying)

A complex (snicker) example: Compute the partial fraction expansion of $\frac{2}{z^2+1}$.

1. Compute the **poles**: $z^2 + 1 = (z - j)(z + j) = 0 \rightarrow p_1 = j$ and $p_2 = -j$
2. Compute the **residues**: $A_1 = \frac{2}{j+j} = -j$ and $A_2 = \frac{2}{-j-j} = j$
3. Plug in: $\frac{2}{z^2+1} = \frac{j}{z+j} - \frac{j}{z-j}$ (confirm by cross-multiplying)
4. Note $p_2 = p_1^*$ and $A_2 = A_1^*$ (conjugate pairs)

The existence of the partial fraction expansion will not be derived here. The formula for the residues follows immediately by multiplying both sides of the expansion by $(z - p_n)$, *performing the cancellation in $X(z)$ and in $\frac{A_n}{z-p_n}$* , and setting $z = p_n$. Then all terms are wiped out except $X(z)(z - p_n)|_{z=p_n} = A_n$.

If $X(z)$ is only proper (i.e., the numerator and denominator polynomials have the same degree), and not strictly proper, then the partial fraction expansion has a constant added to it. If $X(z)$ is improper, it has a polynomial added to its partial fraction expansion. This polynomial can be computed most easily by dividing the denominator into the numerator, but polynomial division is almost a lost art. We won’t use these facts.

Matlab computes the above partial fraction expansion using $[R,P,K]=\text{residue}([1 \ -3], [1 \ -3 \ 2])$; which gives $R=[-1 \ 2]'$ and $P=[2 \ 1]'$, so that Residues -1 and 2 are associated with Poles 2 and 1, respectively. There is no constant K since $\frac{z-3}{z^2-3z+2}$ is strictly proper.

C. Inversion Using Partial Fractions

The following six-step procedure computes the inverse z-transform of a proper rational function $X(z)$:

1. Compute poles $\{p_n\}$ of $X(z)$ by setting denominator polynomial equal to zero and computing roots
2. Write the partial fraction expansion of $X(z)/z$ as:

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \dots + \frac{A_N}{z-p_N} \quad (26)$$

3. Compute the residues $\{A_n\}$ of $X(z)/z$ (NOT $X(z)$) using

$$A_n = \frac{X(z)}{z}(z-p_n)|_{z=p_n} = \frac{(p_n-z_1)\dots(p_n-z_n)\dots(p_n-z_M)}{(p_n-0)(p_n-p_1)\dots(p_n-p_{n-1})(p_n-p_{n+1})\dots(p_n-p_N)} \quad (27)$$

4. Having computed the residues $\{A_n\}$, multiply through by z to get:

$$X(z) = A_0 + A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + \dots + A_N \frac{z}{z-p_N} \quad (28)$$

5. Compute the inverse z-transform $x[n]$ of $X(z)$ by recognition:

$$x[n] = A_0\delta[n] + A_1p_1^n u[n] + A_2p_2^n u[n] + \dots + A_Np_N^n u[n] \quad (29)$$

6. Complex poles and residues occur in complex conjugate pairs. Simplify each pair using the formula:

$$Ap^n + A^*(p^*)^n = 2|A| \cdot |p|^n \cos(\omega t + \theta) \quad \text{where} \quad A = |A|e^{j\theta} \quad \text{and} \quad p = |p|e^{j\omega} \quad (30)$$

In particular, the frequency ω is the angle of the pole. And don't forget that factor of two in the front!

D. Real Simple Example (snicker)

Compute the inverse z-transform of $X(z) = \frac{z-3}{z^2-3z+2}$.

Do you think you can plug in the results of the partial fraction expansion we did above? Guess again:

1. **Compute poles:** $z^2 - 3z + 2 = (z-1)(z-2) = 0 \rightarrow p_1 = 1$ and $p_2 = 2$ as before
2. **Write expansion for $X(z)/z$:** $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-2}$
3. **Compute residues:** $A_0 = \frac{(0-3)}{(0-1)(0-2)} = -\frac{3}{2}$. $A_1 = \frac{(1-3)}{(1-0)(1-2)} = 2$. $A_2 = \frac{(2-3)}{(2-0)(2-1)} = -\frac{1}{2}$.
4. **Multiply by z :** $X(z) = -\frac{3}{2} + 2\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-2}$
5. **Invert by recognition:** $x[n] = -\frac{3}{2}\delta[n] + 2u[n] - \frac{1}{2}(2)^n u[n]$.
6. Complex poles and residues? None here. But see below...

E. Simple Complex Example (snicker)

Compute the inverse z-transform of $X(z) = \frac{2z}{z^2 - 2z + 2}$.

1. **Compute poles:** $z^2 - 2z + 2 = 0 \rightarrow p_1 = 1 + j$ and $p_2 = p_1^* = 1 - j$.
2. **Write expansion for $X(z)/z$:** $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A}{z - (1+j)} + \frac{A^*}{z - (1-j)}$
3. **Compute residues:** $A_0 = \frac{2(0)}{(0 - (1+j))(0 - (1-j))} = 0$. $A_1 = \frac{2(1+j)}{(1+j)((1+j) - (1-j))} = -j$.
4. **Multiply by z :** $X(z) = \frac{-j}{z - (1+j)} + \frac{j}{z - (1-j)}$
5. **Invert by recognition:** $x[n] = -j(1+j)^n u[n] + j(1-j)^n u[n]$.
6. **Simplify complex:** $x[n] = 2(\sqrt{2})^n \sin(\frac{\pi}{4}n)$ since $p = 1 + j = \sqrt{2}e^{j\pi/4}$ and $A = -j = 1e^{-j\pi/2}$.

F. Complex Complex Example (snicker)

Compute the inverse z-transform of $X(z) = \frac{z-1}{z^3 + 4z^2 + 8z + 8}$ (C.T. Chen p. 257).

1. **Compute poles:** $z^3 + 4z^2 + 8z + 8 = 0 \rightarrow p_1 = -2$; $p_2 = 2e^{j2.09}$; $p_2^* = 2e^{-j2.09}$
2. **Write expansion for $X(z)/z$:** $\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-2e^{j2.09}} + \frac{C^*}{z-2e^{-j2.09}}$
3. **Compute residues:**
 - $A = (z-0)\frac{X(z)}{z}|_{z=0} = \frac{0-1}{(0+2)(0-2e^{j2.09})(0-2e^{-j2.09})} = -\frac{1}{8}$
 - $B = (z+2)\frac{X(z)}{z}|_{z=-2} = \frac{-2-1}{(-2)(-2-2e^{j2.09})(-2-2e^{-j2.09})} = \frac{3}{8}$
 - $C = (z-2e^{j2.09})\frac{X(z)}{z}|_{z=2e^{j2.09}} = \frac{(2e^{j2.09}-1)/2e^{j2.09}}{(2e^{j2.09}+2)(2e^{j2.09}-2e^{-j2.09})} = 0.19e^{-j2.29}$
4. **Multiply by z :** $X(z) = -\frac{1}{8} + \frac{3}{8}\frac{z}{z+2} + \frac{0.19e^{-j2.29}}{z-2e^{j2.09}} + \frac{0.19e^{+j2.29}}{z-2e^{-j2.09}}$
5. **Invert by recognition:** $x[n] = -\frac{1}{8}\delta[n] + \frac{3}{8}(-2)^n u[n] + (0.19)2^n e^{j(2.09n-2.29)}u[n] + (0.19)2^n e^{-j(2.09n-2.29)}u[n]$
6. **Simplify complex:** $x(n) = -\frac{1}{8}\delta[n] + \frac{3}{8}(-2)^n u[n] + (0.38)2^n \cos(2.09n - 2.29)u[n]$.

G. Repeated Poles at Origin

There is one case of repeated poles that does arise in practice: repeated poles at $z = 0$ ($z = 0$ is called “the origin” when referring to poles and zeros, since it is the origin of the complex plane).

We already know that dividing its z-transform by z^D merely delays $x[n]$. But we can't just omit a z^D in the denominator, compute the inverse z-transform of the remaining part of $X(z)$, and delay the result, since the remaining $X(z)$ may not be proper. Instead, handle this in the way that the following example shows:

Compute the inverse z-transform of $X(z) = \frac{z^3 + 2z^2 + 3z + 4}{z^2(z-1)}$. Due to the double pole at the origin $z = 0$, we can't use partial fractions here. Instead, rewrite $X(z)$ as

$$X(z) = \frac{z^3 + 2z^2 + 3z + 4}{z^2(z-1)} = \frac{z^3 + 2z^2 + 3z + 4}{z^3} \frac{z}{z-1} = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3}) \frac{z}{z-1} \quad (31)$$

and we recognize the inverse z-transform of this as

$$x[n] = \{ \underline{1}, 2, 3, 4 \} * u[n] = u[n] + 2u[n-1] + 3u[n-2] + 4u[n-3] \quad (32)$$

In general, the numerator divided by z^M becomes a linear combination, which is taken of z^M divided by the denominator. Note $z^M/z^D = z^{M-D}$; if $D > M$ the remainder becomes a delay of $D - M$.

IV. TRANSFER FUNCTIONS

The *Transfer function* of an LTI system is simply the z-transform of its impulse response:

$$\begin{array}{l} \text{TRANSFER} \\ \text{FUNCTION} \end{array} = H(z) = \mathcal{Z}\{h[n]\} = \mathcal{Z}\left\{ \begin{array}{l} \text{IMPULSE} \\ \text{RESPONSE} \end{array} \right\} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \dots \quad (33)$$

That's it?! What use is that? Quite a bit, when you recall that z-transforms convert convolutions to multiplications (the third property of z-transforms above).

A. Input-Output Pairs

Suppose that the response of a given LTI system to input $x[n]$ is output $y[n]$:

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = h[n] * x[n] \rightarrow Y(z) = H(z)X(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} \quad (34)$$

This means that we can compute the transfer function $H(z)$ for an LTI system from *any* input-output pair by simply dividing their z-transforms! We can now solve problems like this:

Given that the response of an LTI system to input $x[n] = (1/2)^{n+1}u[n]$ is $y[n] = \delta[n] - \delta[n-1]$,
compute the *step response*, which is the system's response to a step input $u[n]$
 (compare to *impulse response*, which is the system's response to an impulse $\delta[n]$).

Solution: All we need to do is note that

$$H(z) = \frac{Y(z)}{X(z)} = [1 - z^{-1}] / \left[\frac{1}{2} \frac{z}{z - (1/2)} \right] = (1 - z^{-1})(2 - z^{-1}) = 2 - 3z^{-1} + z^{-2} \rightarrow h[n] = \{\underline{2}, -3, 1\} \quad (35)$$

So from a single input-output pair we have computed transfer function $H(z)$ and impulse response $h[n]$.

Now we could compute the step response using convolution, but that is so last week. Instead, use

$$Y(z) = H(z)X(z) = [2 - 3z^{-1} + z^{-2}] \frac{z}{z-1} = \frac{(2z-1)(z-1)}{z^2} \frac{z}{z-1} = \frac{2z-1}{z} = 2 - z^{-1} \rightarrow y[n] = \{\underline{2}, -1\} \quad (36)$$

Did you guess that the answer would be so simple? Or that the system would “eat” a step function, leaving nothing for $n > 2$? Using transfer functions, we can even see why this happens: The factors $(z-1)$ cancel each other out. If you are observant, you might notice that

$$2u[n] - 3u[n-1] + u[n-2] = 2\delta[n] - \delta[n-1] = \{\underline{2}, -1\} \quad (37)$$

(another case of two different expressions for a signal being equal). But why should you have to be observant, when all you have to do is cancel $(z-1)$? No thinking required (always a good thing).

B. Frequency Response

Transfer functions can do much more than that. Suppose we input a complex exponential $e^{j\omega n}$:

$$x[n] = e^{j\omega n} \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = h[n] * e^{j\omega n} = \sum_{i=0}^{\infty} h[i] e^{j\omega(n-i)} = \sum_{i=0}^{\infty} h[i] e^{-j\omega i} e^{j\omega n} = H(e^{j\omega}) e^{j\omega n} \quad (38)$$

That is, the response to a complex exponential is the complex exponential multiplied by the transfer function evaluated at $z = e^{j\omega}$. This is why I strongly prefer to use $H(e^{j\omega})$ to designate the frequency response function, unlike the “official” course notes (and many textbooks) which use $H(\omega)$. $H(e^{j\omega})$ also serves as a not-so-subtle reminder that discrete-time functions of frequency are all periodic in ω with period 2π .

Continuing with the above example, **compute** the response of the above system to $\cos((\pi/2)n)$. We have

$$H(e^{j\pi/2}) = H(j) = 2 - 3\frac{1}{j} + 1\frac{1}{j^2} = 1 + 3j = \sqrt{10}e^{j1.25} \quad (39)$$

So the response is

$$x[n] = \cos\left(\frac{\pi}{2}n\right) \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = \sqrt{10} \cos\left(\frac{\pi}{2}n + 1.25\right) \quad (40)$$

Going the other way: Compute the transfer function of a system having gain function (magnitude response)

$$|H(e^{j\omega})| = \sqrt{(3 + 4\cos\omega)^2 + 16\sin^2\omega} / \sqrt{(1 + 2\cos\omega)^2 + 4\sin^2\omega} \quad (41)$$

Suppress your first reaction (“How the heck am I supposed to do this?!”) and break the problem down:

- Magnitude of ratio equals ratio of magnitudes, so we can look at numerator and denominator separately
- Square root of the sum of squares of two terms: these must be the real and imaginary parts
- Since $H(e^{j\omega})$ is conjugate symmetric, its real part must be an even function of ω and its imaginary part must be an odd function. So $3 + 4\cos\omega$ is the real part and $\pm 4\sin\omega$ is the imaginary part
- Now it becomes clear: the numerator is $3 + 4e^{j\omega}$ and the denominator is $1 + 2e^{j\omega}$
- So $H(z) = \frac{3+4z}{1+2z}$ is one solution; $\frac{4+3z}{2+z}$, $\frac{3+4z}{2+z}$, $\frac{4+3z}{1+2z}$ are some others. Other possibilities involve sign changes.

V. POLES AND ZEROS

An important way of representing transfer functions deserves its own section.

A. Definition

Let the transfer function of an LTI system be

$$H(z) = \frac{b_0 z^M + \dots + b_M}{a_0 z^N + \dots + a_N} = \frac{b_0(z - z_1) \dots (z - z_M)}{a_0(z - p_1) \dots (z - p_N)} \quad (42)$$

- The **poles** of the system are $\{p_1 \dots p_N\}$
- The **zeros** of the system are $\{z_1 \dots z_M\}$

For example, the transfer function $H(z) = \frac{(z-1)(z-2)}{(z-3)(z-4)}$ has zeros $\{1, 2\}$ and poles $\{3, 4\}$.

“Poles” were already defined in the subsection on partial fraction expansions.

Poles and zeros are typically plotted on a *pole-zero diagram*, in which zeros are designated by O (which makes sense) and poles are designated by \times (which at least can't be confused with O). The *unit circle*

$$\text{UNIT CIRCLE} = \{z : |z| = 1\} = \{z : z = e^{j\omega}\} \quad (43)$$

which is a circle in the complex plane with radius one, is also plotted on pole-zero plots (not to be confused with a super-sized zero O!). Why? See below.

B. Impulse Response

How are poles and zeros any more informative than the coefficients $\{a_n\}$ and $\{b_n\}$?

They are, and in two different ways. The first way has to do with the impulse response of the system:

- The impulse response $h[n] = A_1 p_1^n u[n] + \dots + A_N p_N^n u[n]$
- This means that if $|p_i| > 1$ for **any** i , the system is unstable
- But if $|p_i| < 1$ for **all** i , the system is BIBO stable (see below)
- The closer **any** $|p_i|$ is to one, the longer $h[n]$ takes to decay to zero

If $|p_i| < 1$ for **all** i , then $h[n]$ is absolutely summable since

$$\sum_{n=0}^{\infty} |h[n]| \leq \sum_{n=0}^{\infty} |A_1| \cdot |p_1|^n + \dots + \sum_{n=0}^{\infty} |A_N| \cdot |p_N|^n = \frac{|A_1|}{1 - |p_1|} + \dots + \frac{|A_N|}{1 - |p_N|} \quad (44)$$

We have used the triangle inequality $|x + y| \leq (|x| + |y|)$ for the initial sum and then for each individual sum.

So we have a **simple test for BIBO stability**: *An LTI system is BIBO stable if and only if all of its poles $\{p_i\}$ have $|p_i| < 1$ (strict inequality). That is, all of its poles lie inside the unit circle.*

This explains why the unit circle is plotted on pole-zero plots: To determine whether an LTI system is BIBO stable, all you have to do is see whether all the \times s are inside the large ring!

This also explains why the z-transform is defined using z^{-1} . If we used $X(z) = \sum_{n=0}^{\infty} x[n]z^n$, the condition for BIBO stability would be: all poles lie outside the unit circle. Since stable systems are more interesting than unstable systems (stable systems don't blow up—a good selling point), having their poles inside the unit circle makes for neater pole-zero diagrams.

C. Frequency Response

The other reason has to do with the form of the frequency response. The gain (magnitude of the frequency response function) of a system with transfer function $H(z)$ is

$$|H(e^{j\omega})| = [|e^{j\omega} - z_1| \dots |e^{j\omega} - z_M|] / [|e^{j\omega} - p_1| \dots |e^{j\omega} - p_N|] \quad (45)$$

What does each of these terms contribute to the gain?

Suppose the n^{th} zero $z_n = e^{j\omega_n}$. Then $|e^{j\omega} - z_n| = 0$ at $\omega = \omega_n$. This implies that the gain $|H(e^{j\omega})| = 0$ if there is a zero at $e^{j\omega_0}$ (on the unit circle $|z| = 1$), and $|H(e^{j\omega_0})| \approx 0$ if there's a zero at $Ae^{j\omega_0}$, $A \approx 1$ (near the unit circle). This is where “zeros” get their name—they really do zero out frequencies.

Similarly, let the n^{th} pole be $p_n = Ae^{j\omega_n}$, $A \approx 1$. Then $\frac{1}{|e^{j\omega} - p_n|} = \frac{1}{|A-1|}$ at $\omega = \omega_n$. This implies that the gain $|H(e^{j\omega_0})|$ is large if there is a pole at $Ae^{j\omega_0}$ (near the unit circle). This is where “poles” get their name—they look like tentpoles propping up the gain function (see the plot below). The \times symbol looks like a pole viewed from overhead—all you can see is the top of the pole and four guy wires holding it up.

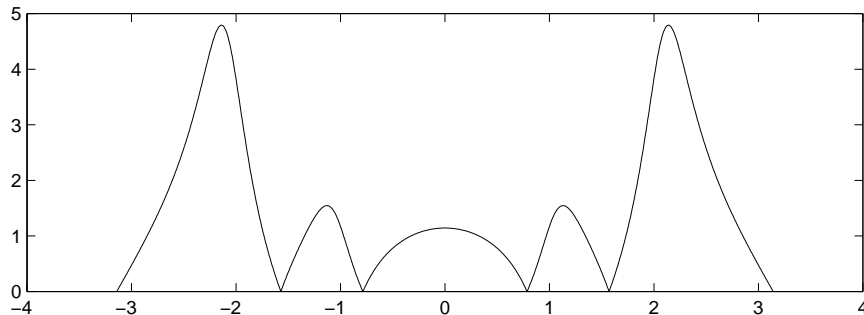
Given the poles $\{p_n\}$ and zeros $\{z_n\}$ near the unit circle, we can plot the gain response roughly as follows:

1. Start on the unit circle at $\omega = 0 \rightarrow z = e^{j\omega} = 1$
2. Trace along the unit circle counterclockwise (increasing ω)
3. When pass a zero at $Ae^{j\omega_m}$, $A \approx 1$, gain dips at ω_m
4. When pass a pole at $Ae^{j\omega_n}$, $A \approx 1$, gain peak at ω_n
5. On unit circle $z = e^{j\omega}$: $z = 1 \rightarrow DC$; $z = j \rightarrow \omega = \frac{\pi}{2}$; $z = -1 \rightarrow \omega = \pi$
6. Due to conjugate symmetry, we can stop when we reach $\omega = \pi \rightarrow z = -1$
7. But note that continuing around the circle will yield an even (symmetric) gain

More importantly, we can use poles and zeros to *design* a filter to have a desired gain response.

As an example, plot the gain function (to a scale factor) for an LTI system having

- **Zeros:** $\{e^{\pm j\pi/4}, e^{\pm j\pi/2}, e^{j\pi}\}$ and **Poles:** $\{0.8e^{\pm j\pi/3}, 0.8e^{\pm j2\pi/3}\}$
- Note gain dips to zero at $\omega = \pm\frac{\pi}{4} = \pm 0.785$ and $\omega = \pm\frac{\pi}{2} = \pm 1.57$
- Note peaks at $\omega = \pm\frac{\pi}{3} = \pm 1.05$ and $\omega = \pm\frac{2\pi}{3} = \pm 2.10$
- The closer a zero or pole is to unit circle, the sharper the peak or dip
- The closer pole is to unit circle, the longer $h[n]$ takes to decay to zero.
- Need all poles inside the unit circle for the system to be BIBO stable



VI. THE ATLANTA AIRPORT

“That’s it; he’s finally lost his mind” (“finally”?) But this analogy does make sense. Read on...

There’s an old saying in the travel industry: “Whether you’re going to heaven or to hell, you’re going to have to transfer in Atlanta.” Indeed, Atlanta is where it is because it was a rail center (it was originally called “Terminus.”) Today, it’s still a rail center, but if you are flying from anywhere in the southern USA to anywhere else in the southern USA, you still have to transfer flights at the Atlanta airport: Take a flight from your starting point to Atlanta, and then take another flight from Atlanta to your destination.

What does this have to do with EECS 206? The transfer function of an LTI system plays the same role as the Atlanta airport in relating different descriptions of the system. To go from any description of an LTI system (impulse response, frequency response, difference equation, etc.) to any other description, first compute the transfer function (fly to Atlanta) then compute the desired description from the transfer function (fly from Atlanta to your destination).

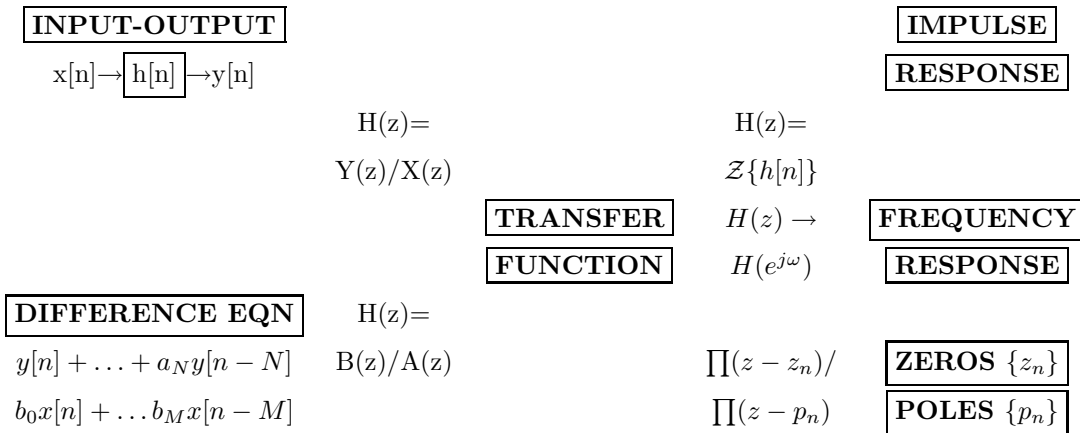
We have already seen an example of this above. From a given input-output pair, we first computed the transfer function, and then *from the transfer function* computed the step, impulse and frequency responses. But starting from **any** description, we can compute **any other** description using the transfer function as an intermediate step. Specifically, the flight schedule to Atlanta is:

- **Impulse Response:** $H(z) = \mathcal{Z}\{h[n]\}$ and $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.
- **Frequency Response:** $H(e^{j\omega}) = H(z)_{z=e^{j\omega}}$ and vice-versa (the frequency response of an LTI system having a finite number of poles and zeros can be written as a function of $e^{j\omega}$ —recall the example above).
- **Poles and Zeros:** $H(z) = \frac{b_0(z-z_1)\dots(z-z_M)}{a_0(z-p_1)\dots(z-p_N)}$ and vice-versa. These specify $H(z)$ to a scale factor b_0/a_0 .
- **Specific input-output pair:** If $x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]$, then $H(z) = Y(z)/X(z)$ and $y[n] = \mathcal{Z}^{-1}\{H(z)X(z)\}$.
- **Difference Equation:** Filters are actually implemented on a computer chip using a *difference equation* (note the analogy to a continuous-time differential equation). A difference equation computes $y[n]$ recursively from past inputs and outputs (and present input) $\{y[n-1] \dots y[n-N], x[n] \dots x[n-M]\}$:

$$a_0y[n] + a_1y[n-1] + \dots + a_Ny[n-N] = b_0x[n] + b_1x[n-1] + \dots + b_Mx[n-M] \quad (46)$$

Given this description, $H(z) = z^{N-M} \frac{b_0z^M + \dots + b_M}{a_0z^N + \dots + a_N}$ and vice-versa. Watch the z^{N-M} in front.

This is all summarized in the figure on the next page:



VII. HALF-DOZEN EXAMPLES

A. Example #1

Problem: Compute poles and zeros of an LTI system having impulse response $h[n] = 2^n u[n] + 4^n u[n]$.

Solution: $H(z) = \frac{z}{z-2} + \frac{z}{z-4} = \frac{2z^2-6z}{z^2-6z+8} \rightarrow$ ZEROS: $\{0,3\}$
 POLES: $\{2,4\}$.

B. Example #2

Problem: Compute poles and zeros of an LTI system $y[n] - 3y[n-1] + 2y[n-2] = x[n] - 4x[n-1]$.

• **Solution:** $H(z) = \frac{1-4z^{-1}}{1-3z^{-1}+2z^{-2}} \frac{z^2}{z^2} = \frac{z^2-4z}{z^2-3z+2}$.

• **Zeros:** Solve $N(z) = z^2 - 4z = 0 \rightarrow z = \{0, 4\}$. A common mistake: Cite only $\{4\}$.

• **Poles:** Solve $D(z) = z^2 - 3z + 2 = 0 \rightarrow z = \{1, 2\}$. Note $z^{N-M} \rightarrow$ zero at the origin.

C. Example #3

Problem: Compute the step response of an LTI system with zero at 1, pole at 3, and $H(0) = 1$.

Solution: $H(z) = 3 \frac{z-1}{z-3}$. $U(z) = \frac{z}{z-1} \rightarrow Y(z) = 3 \frac{z}{z-3} \rightarrow y[n] = 3 \cdot 3^n u[n]$.

Problem: Now compute a difference equation that implements the system.

Solution: You can plug into the formula given above, but it is easier to **cross-multiply**:

$$\frac{Y(z)}{X(z)} = H(z) = 3 \frac{z-1}{z-3} = 3 \frac{1-z^{-1}}{1-3z^{-1}} \rightarrow y[n] - 3y[n-1] = 3x[n] - 3x[n-1].$$

D. Example #4

Problem: Compute $h[n]$ if $x[n] = (-2)^n u[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = \frac{2}{3}(-2)^n u[n] + \frac{1}{3}u[n]$.

Solution: $X(z) = \frac{z}{z+2}$ and $Y(z) = \frac{2z/3}{z+2} + \frac{z/3}{z-1} = \frac{z^2}{(z+2)(z-1)} \rightarrow H(z) = \frac{z}{z-1} \rightarrow h[n] = u[n]$.

The first term of $y[n]$ is the *forced response*, which looks like the input to the system.

The second term of $y[n]$ is the *natural response*, which looks like the impulse response.

Problem: Now compute a difference equation that implements the system.

Solution: Cross-multiply: $\frac{Y(z)}{X(z)} = H(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \rightarrow y[n] - y[n-1] = x[n]$.

This is called an *accumulator* or *summer* system, since it sums up the past input.

Physically, a capacitor would do this (accumulate charge from the current fed into it).

E. Example #5

A problem with real-world applications:

Problem: A cell phone signal $x[n]$ is distorted by multipath reflections off of city buildings.

What your cell phone receives is not $x[n]$ but $y[n]$, where

$$x[n] \rightarrow \boxed{y[n] = x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]} \rightarrow y[n] \quad (47)$$

The goal is to compute the **inverse filter** $g[n]$ that recovers $x[n]$ from $y[n]$:

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \rightarrow \boxed{g[n]} \rightarrow x[n] \quad (48)$$

so that $g[n]$ “undoes” the effect of $h[n]$. How do we do this? *Can* we do this?

Solution: Recall that the overall impulse response of two systems in cascade is the convolution of their impulse responses. So we want $h[n]*g[n] = \delta[n]$, which implies $H(z)G(z) = 1$. For our problem, this becomes

$$\begin{aligned} h[n] = \{1, -\frac{3}{4}, \frac{1}{8}\} &\rightarrow H(z) = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = (z^2 - \frac{3}{4}z + \frac{1}{8})/z^2 \\ &\rightarrow G(z) = 1/H(z) = z^2/[z^2 - \frac{3}{4}z + \frac{1}{8}] = z^2/[(z - \frac{1}{2})(z - \frac{1}{4})] \end{aligned} \quad (49)$$

Now that we’re at the Atlanta airport (we have the transfer function $G(z)$), we know what to do:

$$\frac{G(z)}{z} = \frac{z}{(z - 1/2)(z - 1/4)} = \frac{2}{z - 1/2} - \frac{1}{z - 1/4} \rightarrow G(z) = 2\frac{z}{z - 1/2} - 1\frac{z}{z - 1/4}$$

$$\text{Residues: } 2 = (1/2)/[(1/2) - (1/4)]; \quad -1 = (1/4)/[(1/4) - (1/2)] \rightarrow g[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n] \quad (50)$$

Note that the inverse filter is BIBO stable since the zeros of $H(z)$ =poles of $G(z)$ are inside the unit circle.

Also note that $g[0] \neq 0$ since the numerator and denominator of $G(z)$ have the same degrees.

F. Example #6

We end this document with the problem at the beginning of this document:

Given that the response of an LTI system to input $(\frac{1}{2})^n u[n]$ is output $\{0, 0, 1\} = \delta[n - 2]$,

Compute the response of the system to input $2 \cos(\frac{\pi}{3}n)$. That is:

If $x[n] = (\frac{1}{2})^n u[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = \{0, 0, 1\}$, then $x[n] = 2 \cos(\frac{\pi}{3}n) \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = ?$

$$\begin{aligned} H(z) &= \frac{\mathcal{Z}\{y[n]\}}{\mathcal{Z}\{x[n]\}} = z^{-2} / \frac{z}{z - (1/2)} = \frac{z - (1/2)}{z^3}. \quad H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = (e^{j\omega} - (1/2))/e^{j3\omega} \\ H(e^{j\pi/3}) &= [e^{j\pi/3} - (1/2)]/e^{j3\pi/3} = [(1/2) + j\sqrt{3}/2 - (1/2)]/(-1) = -j\sqrt{3}/2 = (\sqrt{3}/2)e^{-j\pi/2} \\ 2 \cos((\pi/3)n) &\rightarrow \boxed{\text{LTI}} \rightarrow \sqrt{3} \cos((\pi/3)n - \pi/2) = \sqrt{3} \sin((\pi/3)n). \quad \text{Also, } h[n] = \{0, 0, 1, -(1/2)\} \end{aligned} \quad (51)$$