TRANSFER (ALSO KNOWN AS SYSTEM) FUNCTIONS

DEF: $H(z) = N(z)/D(z) = Y(z)/X(z) = B(z)/A(z)$ as defined below.

Poles: Roots of $D(z) = 0$. **ARMA:** $\sum a[i]y[n-i] = \sum b[j]x[n-j]$.

Zeros: Roots of $N(z) = 0$. **Impulse response:** $\delta[n] \to \boxed{h[n]} \to h[n]$.

Transfer functions are associated with zero initial conditions.

EX #1: $h[n] = 2^n u[n] + 4^n u[n] \to H(z) = \frac{z}{z-2} + \frac{z}{z-4} = \frac{2z^2-6z}{z^2-6z+8}$. ZEROS: {0,3} POLES: {2,4}.

EX #2: $y[n] - 3y[n-1] + 2y[n-2] = x[n] - 4x[n-1] \to H(z) = \frac{1-4z^{-1}}{1-3z^{-1}+2z^{-2}}$.

where: $A(z) = \mathcal{Z}\{1,-3,2\} = 1-3z^{-1}+2z^{-2}$ and $B(z) = \mathcal{Z}\{1,-4\} = 1-4z^{-1}$.

z not z^{-1}: $H(z) = \frac{1-4z^{-1}}{1-3z^{-1}+2z^{-2}} \frac{z^2}{z^2} = \frac{z^2-4z}{z^2-3z+2}$. Use **this** to compute poles+zeros.

Zeros: Solve $N(z) = z^2 - 4z = 0 \to z = \{0, 4\}$. Common mistake: cite only 4.

Poles: Solve $D(z) = z^2 - 3z + 2 = 0 \to z = \{1, 2\}$. Use Matlab's **roots**.

Note: $H(z) = 0$ for z=any zero while $H(z) \to \infty$ for z=any pole.

There are 5 ways to describe an LTI system: (1) $h[n]$; (2) $H(z)$; (3) Difference equation (4) Any input-output pair (5) Poles+zeros. Using transfer functions, **can go from any of these to any other:**

INPUT–OUTPUT: Input $x[n] \to \boxed{h[n]} \to$output $y[n]$ $\underbrace{\Longleftrightarrow}_{Y(z)/X(z)}$ $\mathbf{H(z)}$ $\underbrace{\Longleftrightarrow}_{\mathcal{Z}\{h[n]\}}$ $h[n] =$ IMPULSE RESPONSE

ARMA: $\sum a[i]y[n-i] = \sum b[j]x[n-j]$ $\underbrace{\Longleftrightarrow}_{B(z)/A(z)}$ $\mathbf{H(z)}$ $\underbrace{\Longleftrightarrow}_{C\prod \frac{z-z_i}{z-p_i}}$ **POLE ZERO** plot.

EX #3: $x[n] = (-2)^n u[n] \to \boxed{h[n]} \to y[n] = \frac{2}{3}(-2)^n u[n] + \frac{1}{3}u[n]$. Find $h[n]$.

Soln: $X(z) = \frac{z}{z+2}$. $Y(z) = \frac{2z/3}{z+2} + \frac{z/3}{z-1} = \frac{z^2}{(z+2)(z-1)}$. $H(z) = \frac{z}{z-1}$. $h[n] = u[n]$.

First term of $y[n]$=forced response. Second term=natural response.

Difference eqn.: $\frac{Y(z)}{X(z)} = H(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \to y[n] - y[n-1] = x[n]$.

EX #4: Find difference equation implementing $H(z) = \frac{z^3+7z^2}{z^3-z^2+2z-3}$.

Soln: Write $H(z) = \frac{Y(z)}{X(z)} = \frac{1+7z^{-1}}{1-z^{-1}+2z^{-2}-3z^{-3}}$ and **cross-multiply**:

$Y(z)(1 - z^{-1} + 2z^{-2} - 3z^{-3}) = X(z)(1 + 7z^{-1}) \to$difference equation:

$\mathcal{Z}^{-1}$: $y[n] - y[n-1] + 2y[n-2] - 3y[n-3] = x[n] + 7x[n-1]$.

EX #5: Find step response of system with zero at 1, pole at 3, and $H(0) = 1$.

Soln: $H(z) = 3\frac{z-1}{z-3}$. $U(z) = \frac{z}{z-1} \to Y(z) = 3\frac{z}{z-3} \to y[n] = 3 \cdot 3^n u[n]$.

$\frac{Y(z)}{X(z)} = H(z) = 3\frac{z-1}{z-3} = 3\frac{1-z^{-1}}{1-3z^{-1}} \to y[n] - 3y[n-1] = 3x[n] - 3x[n-1]$.

POLES & ZEROS AND FREQUENCY RESPONSE

Given: Locations of zeros $\{z_1 \ldots z_M\}$ and poles $\{p_1 \ldots p_N\}$ of a filter.
Goal: *Shape* of gain function=$|H(\omega)|$=magnitude of frequency response.

H(z): $H(z) = \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$ (ignore any constant in front).
H(w): $|H(\omega)| = [|e^{j\omega} - z_1| \ldots |e^{j\omega} - z_M|]/[|e^{j\omega} - p_1| \ldots |e^{j\omega} - p_N|]$.
What does each of these terms contribute to gain $|H(\omega)|$?

Zeros: Let n^{th} zero $z_n = e^{j\omega_n}$. Then $|e^{j\omega} - z_n| = 0$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)| = 0$ if there is a zero at $e^{j\omega_o}$ (on the unit circle $|z| = 1$).
- Gain $|H(\omega_o)| \approx 0$ if there's a zero at $Ae^{j\omega_o}, A \approx 1$ (near the unit circle).

Poles: Let n^{th} pole $p_n = Ae^{j\omega_n}, A \approx 1$. Then $\frac{1}{|e^{j\omega} - p_n|} = \frac{1}{|A-1|}$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)|$ is large if there is a pole at $Ae^{j\omega_o}$ (near $|z| = 1$).

1. Start on the unit circle at $\omega = 0 \to z = e^{j\omega} = 1$.
2. Trace along the unit circle counterclockwise (increasing ω)
3. When pass a zero at $Ae^{j\omega_m}, A \approx 1$, gain dips at ω_m.
4. When pass a pole at $Ae^{j\omega_n}, A \approx 1$, gain peak at ω_n.
5. On unit circle $z = e^{j\omega}$: $z = 1 \to DC; z = j \to \omega = \frac{\pi}{2}; z = -1 \to \omega = \pi$.

EX: **Zeros:** $\{e^{\pm j\pi/4}, e^{\pm j\pi/2}, e^{j\pi}\}$. **Poles:** $\{0.8e^{\pm j\pi/3}, 0.8e^{\pm j2\pi/3}\}$. Gain:

1. Note gain dips to zero at $\omega = \pm\frac{\pi}{4} = \pm 0.785$ and $\omega = \pm\frac{\pi}{2} = \pm 1.57$.
2. Note peaks at $\omega = \pm\frac{\pi}{3} = \pm 1.05$ and $\omega = \pm\frac{2\pi}{3} = \pm 2.10$
3. The closer a zero or pole is to unit circle, the sharper the peak or dip.
4. The closer pole is to unit circle, the longer $h[n]$ takes to decay to zero.
5. Need all poles inside the unit circle for the system to be BIBO stable.