CONTINUOUS-TIME AND DISCRETE-TIME SINUSOIDS

DEF: A **sinusoidal signal** has the standard form $x(t) = A\cos(\omega_o t + \phi)$.
Where: $A \geq 0$=**amplitude**; $\omega_o \geq 0$=**frequency**; $|\phi| \leq \pi$=**phase** (shift).
Freq: $f_o = \omega_o/(2\pi)$=circular or cyclic frequency in Hertz=cycles per second.
Note: Sinusoid $x(t) = A\cos(2\pi f_o t + \phi)$ has cyclic frequency f_o Hertz.
Period: $T = \frac{1}{f_o} = \frac{2\pi}{\omega_o} \to x(t) = x(t+T)$ **for all** t using $\cos(x) = \cos(x + 2k\pi)$.
DC: DC (constant) signals: $\omega_o = f_o = 0$; $T = \infty$ (low-frequency limit).
RMS(x)= $\sqrt{\frac{1}{T}\int_0^T A^2\cos^2(\omega_o t + \phi)dt} = \sqrt{\frac{1}{T}\int_0^T \frac{A^2}{2}(1 + \cos(2\omega_o t + 2\phi))dt} = \frac{A}{\sqrt{2}}$
using $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and $M(\cos(t)) = 0$ (by inspection).
DC: $\omega_o = 0 \to x(t) = A\cos\phi \to RMS(x) = A|\cos\phi| \geq 0$ (note $A \geq 0$).

Note: Can convert other forms to this standard form. **Some examples:**

1. $x(t) = -3\cos(7t + 0.1) = 3\cos(7t \pm \pi + 0.1)$ using $-\cos(x) = \cos(x \pm \pi)$.
2. $x(t) = 3\sin(7t + 0.1) = 3\cos(7t - \frac{\pi}{2} + 0.1)$ using $\sin(x) = \cos(x - \frac{\pi}{2})$.
3. $x(t) = -3\cos(-7t + 0.1) = 3\cos(7t \pm \pi - 0.1)$ using $\cos(-x) = \cos(x)$.

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x(t): Continuous-time sinusoids are periodic with period $T = \frac{1}{f_o} = \frac{2\pi}{\omega_o}$.
x[n]: Discrete-time sinusoids are **NOT** periodic **UNLESS** f_o is rational.
$\omega_o = 2\pi\frac{K}{T} \Leftrightarrow x[n]$ has period T, provided $\frac{K}{T}$ is reduced to lowest terms.
Why? $A\cos(\omega_o n + \phi) = A\cos(\omega_o(n + T) + \phi) \Leftrightarrow \omega_o T = 2\pi k \Leftrightarrow f_o = \frac{\omega_o}{2\pi} = \frac{k}{T}$.
EX: $\omega_o = 2\pi \to T = 1$; $\omega_o = (2.02)\pi \to T = 100$; $\omega_0 = 2 \to$nonperiodic.

Freq.: $A\cos(\omega_o n + \phi) = A\cos([\omega_o + 2k\pi]n + \phi) \to \omega_o$ **itself** is periodic.
As ω_o increases from 0 to π, oscillation rate increases.
As ω_o increases from π to 2π, oscillation rate decreases.
Huh? Frequency range $\pi \leq \omega_o \leq 2\pi \Leftrightarrow -\pi \leq \omega_o \leq 0$ since ω_o periodic.
WLOG: Restrict discrete-time frequencies to range $-\pi \leq \omega_o \leq \pi$.
Note: $\omega_o = 0 \to \cos(\omega_o n) = 1$; $\omega_o = \pm\pi \to \cos(\omega_o n) = (-1)^n$.

SUM OF MULTIPLE SINUSOIDS

If: $x(t)$ has period T_x and freq. f_x; $y(t)$ has period T_y and freq. f_y;
And: T_x/T_y is a **rational number** (otherwise $x(t) + y(t)$ not periodic);
Then: $x(t) + y(t)$ has period T_{x+y}=Least Common Multiple of T_x and T_y;
Then: $x(t) + y(t)$ has freq. f_{x+y}=Greatest Common Divisor of f_x and f_y.
Note: Always works for sinusoids; rarely fails for general periodic signals.
How? (1) Reduce $\frac{T_x}{T_y} = \frac{M}{N}$ to lowest terms. (2) $T_{x+y} = NT_x = MT_y$.
Why? $\frac{T_x}{T_y} = \frac{M}{N} \to x(t + T_{x+y}) + y(t + T_{x+y}) = x(t + NT_x) + y(t + MT_y)$.
EX: $x(t) = \cos(0.3\pi t) + \cos(\frac{2}{3}\pi t) \to f_o = \frac{3}{20}, \frac{1}{3} \to f_{x+y} = \frac{1}{60} \to T_{x+y} = 60$.

ELEMENTARY SIGNAL OPERATIONS: TRANSLATION

Value: $y(t) = x(t) + c$ **shifts** plot of $x(t)$ **up** by c if $c > 0$ (down if $c < 0$).
Value: $y(t) = cx(t)$ **scales** vertical axis of plot by factor c.

Time: $y(t) = x(t - d)$ **shifts** plot of $x(t)$ **right** by d if $d > 0$ (left if $d < 0$).
Time: $y(t) = x(dt)$ **scales** horizontal axis of plot by factor d.
Time: $y(t) = x(-t)$ **reverses**: flips plot about $t = 0$ axis.
Note: For discrete time signals $x[n]$, d must be an integer!

EX #1: $y(t) = x(t - 2)$ **delays** $x(t)$ by 2: $y(2) = x(0), y(3) = x(1)$.
EX #2: $y(t) = x(2t)$ **shrinks** $x(t)$ by 2: $y(1) = x(2), y(2) = x(4)$.
Means: If $x(t) = \cos(\omega_o t)$, this doubles frequency & halves period.
EX #3: $y(t) = x(-t)$: If $x(t)$ audio, then $y(t)$ is $x(t)$ **backwards**!
Play Beatles records backwards to find hidden messages? Doppler.

COMBINING SIGNAL OPERATIONS

- One of the trickiest topics in EECS 206! Be very careful.
- Two methods are available. Pick one and stick with it.

Why? See lecture notes. Reread at least three times.

- **First Method: Shift then Scale**

1. Put problem into the form $y(t) = x(at - b)$.
2. Shift $x(t)$ by b. $b > 0 \rightarrow$shift right. $b < 0 \rightarrow$shift left.
3. Scale result of #2 in time by a. $a > 1 \rightarrow$compress. $0 < a < 1 \rightarrow$expand.
   $a < 0 \rightarrow$time reversal: flip plot around $t = 0$ axis.

- **Second Method: Scale then Shift**

1. Put problem into the form $y(t) = x(c(t - d))$.
2. Scale $x(t)$ in t by c. $c > 1 \rightarrow$compress in time. $0 < c < 1 \rightarrow$expand.
   $c < 0 \rightarrow$time reversal: flip plot around $t = 0$ axis.
3. Shift result of #2 in time by d. $d > 0 \rightarrow$shift right. $d < 0 \rightarrow$shift left.

EX: $x(t) = 5\cos(3t + 2)$. $y(t) = 4x(2t - 1)$. $y(3) = 4x(6 - 1) = 20\cos(17)$.
Result: $y(t) = 20\cos(3(2t - 1) + 2) = 20\cos(6t - 1)$. Frequency doubles.

DEF: **Linear combination** $z(t)$ of $x(t)$ and $y(t)$:
$z(t) = ax(t) + by(t)$ for two **constants** a and b.
NOT: $tx(t) + 3y(t)$ is NOT a linear combination of $x(t)$ and $y(t)$.
EX: Audio mixing; summing Fourier series (later in EECS 206).
DEF: **Concatenation** $z(t)$ of finite-support $x(t)$ and $y(t)$:
$x(t)$ has support $[0, T_x] \rightarrow z(t) = x(t)$ for $0 \leq t \leq T_x$.
$y(t)$ has support $[0, T_y] \rightarrow z(t) = y(t - T_x)$ for $T_x \leq t \leq (T_x + T_y)$.
$z(t)$ has support $[0, T_x + T_y] \Leftrightarrow z(t) = 0$ for $t < 0$ or $t > (T_x + T_y)$.