EXAMPLES OF DISCRETE-TIME SIGNALS

1. $\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$ (impulse); $u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ (step).
2. $a = |a|e^{j\omega_o}; c = |c|e^{j\theta_o} \rightarrow ca^n = |c||a|^n[\cos(\omega_o n + \theta_o) + j\sin(\omega_o n + \theta_o)]$.
3. **Given x[n]:** $x[n] = x_e[n] + x_o[n]$=even part+odd part
   where $x_e[n] = \frac{1}{2}(x[n] + x[-n])$ and $x_o[n] = \frac{1}{2}(x[n] - x[-n])$.
4. $x[n] = \{3, \underline{1}, 4, 6\} \Leftrightarrow x[n] = 3\delta[n+1] + 1\delta[n] + 4\delta[n-1] + 6\delta[n-2]$.
5. $x[n] = \sum_i x[n]\delta[n-i] = x[n] * \delta[n]$ (sifting property of impulse).

Delay: $x[n-D]$ is $x[n]$ shifted *right* (later) if $D > 0$; *left* (earlier) if $D < 0$.
Fold: $x[-n]$ is $x[n]$ flipped/folded/reversed around $n = 0$ (vertical axis).
Both: $x[N-n]$ is $x[-n]$ shifted *right* if $N > 0$ (since $x[0]$ is now at $n = N$).

DISCRETE-TIME SINUSOIDS

Cont.: $x(t) = A\cos(\omega_o t + \phi)$ has amplitude=A, period=$T = \frac{2\pi}{\omega_o}$, phase=$\phi$
Disc.: $x[n] = A\cos(\omega_o n + \phi)$ (n is an integer) **isn't even periodic unless** $\frac{\omega_o}{2\pi} = f_o = \frac{K}{T}$ is rational. Period=T if fraction $\frac{K}{T}$ lowest terms.

Freq.: $A\cos(\omega_o n + \phi) = A\cos([\omega_o + 2k\pi]n + \phi) \rightarrow \omega_o$ **itself** is periodic.
As ω_o increases from 0 to π, oscillation rate increases.
As ω_o increases from π to 2π, oscillation rate decreases.
Huh? Frequency range $\pi \leq \omega_o \leq 2\pi \Leftrightarrow -\pi \leq \omega_o \leq 0$ since ω_o periodic.
WLOG: Restrict discrete-time frequencies to range $-\pi \leq \omega_o \leq \pi$.
Note: $\omega_o = 0 \rightarrow \cos(\omega_o n) = 1; \quad \omega_o = \pm\pi \rightarrow \cos(\omega_o n) = (-1)^n$.

SAMPLING AND ALIASING

EX #1: $x(t) = \sin(t)$ has maximum frequency 1 $\frac{\text{RADIAN}}{\text{SECOND}}$.
But: *Sample* every $\Delta = \pi \rightarrow x[n] = x(t = n\pi) = \sin(n\pi) = 0$!

EX #2: $x(t) = \cos(2\pi t) + 2\cos(6\pi t)$ (1 Hz & 3 Hz).
But: *Sample* every $\Delta = \frac{1}{5}$ (5 Hz sampling rate)
$\rightarrow x[n] = x(t = \frac{n}{5}) = \cos(\frac{2\pi}{5}n) + 2\cos(\frac{6\pi}{5}n) = \cos(\frac{2\pi}{5}n) + 2\cos(\frac{4\pi}{5}n)$.
Huh? $y(t) = \cos(2\pi t) + 2\cos(4\pi t)$ sampled at 5 Hz→**same** *sampled* signal!
The 3 Hz sinusoid is **aliased** to a 2 Hz sinusoid by sampling at 5 Hz.

Fold: *Original* 3 Hz *folded* across 5/2=2.5 Hz becomes *aliased* 2 Hz.
5/2=**folding** frequency since freqs. above it are folded along it.

Q: How can you tell whether a sampled signal is aliased?
A: Increase the sampling rate as much as possible. Then:

- If the form of the signal doesn't change, it is not aliased.
- If *reduce* sampling rate, at some freq. the *form* of the signal changes.
  That is the *Nyquist sampling rate.* Often a good idea to *oversample.*

QUANTIZATION OF SIGNALS: DIGITAL VS. DISCRETE-TIME:

Quantize the *sampled* $x[n] = x(t = n\Delta)$ (see over for Δ) to 2^B *levels*:

- Let $\delta = (MAX(x[n]) - MIN(x[n]))/2^B$ for quantization using B bits.
- Then for each n: $\hat{x}[n] = MIN(x[n])$ OR $\hat{x}[n] = MIN(x[n]) + \delta$ OR...
- OR $\hat{x}[n] = MAX(x[n]) - \delta$ OR $\hat{x}[n] = MAX(x[n]) \sim$ histogram bins.

- **Note:** Sampling is invertible: Can reconstruct $x(t)$ from $x[n]$
- Quantization **not** invertible: Can't reconstruct $x[n]$ from $\hat{x}[n]$.
- MSE$=\frac{1}{N+1}\sum_{n=0}^{N}(x[n]-\hat{x}[n])^2 = \frac{\delta^2}{12} = \frac{1}{12}(MAX(x[n])-MIN(x[n]))^2\frac{1}{2^{2B}}$.

- $-A \leq x[n] \leq A \rightarrow \delta = \frac{2A}{2^B} \rightarrow MSE = \frac{\delta^2}{12} = \frac{A^2/3}{2^{2B}}$. For sinusoids:
- Signal-to-Quantization-Noise Ratio=SQNR$=\frac{A^2/2}{MSE} = \frac{3}{2}2^{2B}$.

- Let a digital communication link be used to transmit a signal $x(t)$.

Q: Continuous signal $x(t)$ is 8-bit quantized and sampled at 8192 Hz.

A: Bit rate=(8 bits/sample)(8192 samples/sec)=65536 bps (bits/sec).