BASIC SYSTEM PROPERTIES

What: Input $x[n] \rightarrow \overline{|\text{SYSTEM}|} \rightarrow y[n]$ output.

Why: **Design** the system to **filter** input $x[n]$.

DEF: A system is **LINEAR** if these two properties hold:

1. **Scaling:** If $x[n] \rightarrow \overline{|\text{SYS}|} \rightarrow y[n]$, then $ax[n] \rightarrow \overline{|\text{SYS}|} \rightarrow ay[n]$

for: any **constant** a. NOT true if a varies with time (i.e., $a[n]$).

2. **Superposition:** If $x_1[n] \rightarrow \overline{|\text{SYS}|} \rightarrow y_1[n]$ and $x_2[n] \rightarrow \overline{|\text{SYS}|} \rightarrow y_2[n]$,

Then: $(ax_1[n] + bx_2[n]) \rightarrow \overline{|\text{SYS}|} \rightarrow (ay_1[n] + by_2[n])$

for: any **constants** a, b. NOT true if a or b vary with time (i.e., $a[n], b[n]$).

EX: $y[n] = 3x[n-2]$; $y[n] = x[n+1] - nx[n] + 2x[n-1]$; $y[n] = \sin(n)x[n]$.

NOT: $y[n] = x^2[n]$; $y[n] = \sin(x[n])$; $y[n] = |x[n]|$; $y[n] = x[n]/x[n-1]$.

NOT: $y[n] = x[n] + 1$ (try it). This is called an **affine** system.

HOW: If any nonlinear function of $x[n]$, not linear. Nonlinear of just n OK.

DEF: A system is **TIME-INVARIANT** if this property holds:

If $x[n] \rightarrow \overline{|\text{SYS}|} \rightarrow y[n]$, then $x[n-N] \rightarrow \overline{|\text{SYS}|} \rightarrow y[n-N]$

for: any integer time delay N. NOT true if N varies with time (e.g., $N(n)$).

EX: $y[n] = 3x[n-2]$; $y[n] = \sin(x[n])$; $y[n] = x[n]/x[n-1]$.

NOT: $y[n] = nx[n]$; $y[n] = x[n^2]$; $y[n] = x[2n]$; $y[n] = x[-n]$.

HOW: If n appears anywhere other than in $x[n]$, not time-invariant. Else OK.

DEF: A system is **CAUSAL** if it has this form for some function $F(\cdot)$:
$y[n] = F(x[n], x[n-1], x[n-2] \ldots)$ (present and past input only).

Note: Physical systems must be causal. But DSP filters need not be causal!

DEF: A system is **MEMORYLESS** if $y[n] = F(x[n])$ (present input only).

DEF: A system is **(BIBO) STABLE** iff: Let $x[n] \rightarrow \overline{|\text{SYSTEM}|} \rightarrow y[n]$.

If $|x[n]| < M$ for some constant M, then $|y[n]| < N$ for some N.

i.e.: "Every bounded input (BI) produces a bounded output (BO)."

HOW: BIBO stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < L$ for some constant L

where: Impulse $\delta[n] \rightarrow \overline{|\text{SYS}|} \rightarrow h[n]$=impulse response.

EX: A time-invariant system is observed to have these two responses:
$\{\underline{0}, 0, 3\} \rightarrow \overline{|\text{SYS}|} \rightarrow \{\underline{0}, 1, 0, 2\}$ and $\{\underline{0}, 0, 0, 1\} \rightarrow \overline{|\text{SYS}|} \rightarrow \{1, \underline{2}, 1\}$.

Prove: The system is nonlinear.

Proof: By contradiction. Suppose the system is linear. But then:
$\{\underline{0}, 0, 0, 1\} \rightarrow \overline{|\text{SYS}|} \rightarrow \{1, \underline{2}, 1\}$ implies $\{\underline{0}, 0, 3\} \rightarrow \overline{|\text{SYS}|} \rightarrow \{3, 6, \underline{3}\}$
since we know it is time-invariant. Then $\{\underline{0}, 0, 3\}$ produces two outputs!

CONVOLUTION AND IMPULSE RESPONSE

$x[n] = \{3, \underline{1}, 4, 6\} \Leftrightarrow x[n] = 3\delta[n+1] + 1\delta[n] + 4\delta[n-1] + 6\delta[n-2]$.

Note: $x[n] = \sum_i x[i]\delta[n-i] = x[n] * \delta[n]$ (sifting property of impulse).

Delay: $x[n-D]$ is $x[n]$ shifted *right* (later) if $D > 0$; *left* (earlier) if $D < 0$.

Fold: $x[-n]$ is $x[n]$ flipped/folded/reversed around $n = 0$.

Both: $x[N-n]$ is $x[-n]$ shifted *right* if $N > 0$ (since $x[0]$ is now at $n = N$).

FOR LINEAR TIME-INVARIANT (LTI) SYSTEMS:

1. $\delta[n] \rightarrow \boxed{LTI} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \rightarrow \boxed{LTI} \rightarrow h[n-i]$ **Time invariant:** delay by i.
3. $x[i]\delta[n-i] \rightarrow \boxed{LTI} \rightarrow x[i]h[n-i]$ **Linear:** scale by $x[i]$.
4. $\sum_i x[i]\delta[n-i] \rightarrow \boxed{LTI} \rightarrow \sum_i x[i]h[n-i]$ **Linear:** superposition.
5. $x[n] \rightarrow \boxed{LTI} \rightarrow y[n] = \sum_i x[i]h[n-i] = h[n] * x[n]$ **Convolution**.
   Input $x[n]$ into LTI system with no initial stored energy$\rightarrow$output $y[n]$.

PROPERTIES OF DISCRETE CONVOLUTION

1. $y[n] = h[n] * x[n] = x[n] * h[n] = \sum_i h[i]x[n-i] = \sum h[n-i]x[i]$.
2. $h[n], x[n]$ both causal ($h[n] = 0$ for $n < 0$ and $x[n] = 0$ for $n < 0$)
   $\rightarrow y[n] = \sum_{i=0}^{n} h[i]x[n-i] = \sum_{i=0}^{n} h[n-i]x[i]$ also causal.

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3. Suppose $h[n] \neq 0$ *only for* $0 \leq n \leq L$ ($h[n]$ has length $L+1$).
   Suppose $x[n] \neq 0$ *only for* $0 \leq n \leq M$ ($x[n]$ has length $M+1$).
   Then $y[n] \neq 0$ only for $0 \leq n \leq L+M$ ($y[n]$ has length $L+M+1$).

Note: Length[$y[n]$]=Length[$h[n]$]+Length[$x[n]$]-1.

Note: $y[0] = h[0]x[0]$; $\quad y[L+M] = h[L]x[M]$; $\quad x[n] * \delta[n-D] = x[n-D]$.

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4. $x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$ (cascade connection)
   Equivalent to: $x[n] \rightarrow \boxed{h_1[n] * h_2[n]} \rightarrow y[n]$.

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5. $x[n] \rightarrow \begin{array}{c} \nearrow \rightarrow \boxed{h_1[n]} \rightarrow \searrow \\ \searrow \rightarrow \boxed{h_2[n]} \rightarrow \nearrow \end{array} \bigoplus \rightarrow y[n]$ (parallel connection)
   Equivalent to: $x[n] \rightarrow \boxed{h_1[n] + h_2[n]} \rightarrow y[n]$.

MA: $y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_q x[n-q]$ (Moving Average)

Huh? Present output=weighted average of q *most recent* inputs.

Note: Equivalent to $y[n] = b[n] * x[n]$ where $b[k] = b_k, 0 \leq k \leq q$.

FIR: Finite Impulse Response $\Leftrightarrow h[n]$ has finite *duration*.

EX: Any MA system is also an FIR system, and vice-versa.

IIR: Infinite Impulse Response $\Leftrightarrow h[n]$ not finite duration.

EX: $h[n] = a^n u[n] = a^n$ for $n \geq 0$ and $|a| < 1$ is stable and IIR.