INVERSE Z-TRANSFORMS

Given: $X(z)=\frac{b_0+b_1z+\ldots+b_Mz^M}{a_0+a_1z+\ldots+a_Nz^N}$. In EECS 206: Always have $M\le N$.
$\frac{X(z)}{z}=\frac{b_0+b_1z+\ldots+b_Mz^M}{a_0z+a_1z^2+\ldots+a_Nz^{N+1}}=$ RATIO OF TWO POLYNOMIALS $=$ RATIONAL FUNCTION.

Poles: $\{0,p_1\ldots p_N\}$ are roots of $a_0z+\ldots+a_Nz^{N+1}=0$; assume p_n distinct.
Compute using "roots" in Matlab. a_n real$\rightarrow$complex conjugate pairs.

Partial fraction expansion $\frac{X(z)}{z}=\frac{A_0}{z}+\frac{A_1}{z-p_1}+\ldots+\frac{A_N}{z-p_N}$ since distinct poles: $0\neq p_1\neq\ldots\neq p_N$.
$A_n=(z-p_n)X(z)/z$ evaluated at $z=p_n$. OR: "residuez" in Matlab.
NOTE: $p_{n+1}=p_n^*\rightarrow A_{n+1}=A_n^*$: Coeffs also complex conjugate pairs.

Causal signal $X(z)=A_0+A_1\frac{z}{z-p_1}+\ldots+A_N\frac{z}{z-p_N}$. Term-by-term, compute $\mathcal{Z}^{-1}$:
$x[n]=A_0\delta[n]+A_1p_1^nu[n]+\ldots+A_Np_N^nu[n]$ is sum of geometric signals.

Complex: conjugate $Ap^n+A^*(p^*)^n=2|A||p|^n\cos(\omega_0n+\theta)$ where $A=|A|e^{j\theta}; p=|p|e^{j\omega_0}$.
This is *much* easier than trying to use sines and cosines directly!

EX #1: *Simple real example:* Compute inverse z-xform of $X(z)=\frac{z-3}{z^2-3z+2}$.

1. Write $\frac{X(z)}{z}=\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_2}{z-2}$ since $z^2-3z+2=(z-1)(z-2)$.
2. $A_0=\frac{(0-3)}{(0-1)(0-2)}=-\frac{3}{2}$. $A_1=\frac{(1-3)}{(1-0)(1-2)}=2$. $A_2=\frac{(2-3)}{(2-0)(2-1)}=-\frac{1}{2}$.
3. $X(z)=-\frac{3}{2}+2\frac{z}{z-1}-\frac{1}{2}\frac{z}{z-2}\rightarrow x[n]=-\frac{3}{2}\delta[n]+2u[n]-\frac{1}{2}(2)^nu[n]$.

EX #2: *Simple complex example:* Compute inverse z-xform of $X(z)=\frac{2z}{z^2-2z+2}$.

1. $\frac{X(z)}{z}=\frac{A_0}{z}+\frac{A_1}{z-(1+j)}+\frac{A_1^*}{z-(1-j)}$ since $z^2-2z+2=(z-(1+j))(z-(1-j))$.
2. $A_0=\frac{2(0)}{(0-(1+j))(0-(1-j))}=0$. $A_1=\frac{2(1+j)}{(1+j)((1+j)-(1-j))}=-j$.
   $X(z)=\frac{-j}{z-(1+j)}+\frac{j}{z-(1-j)}\rightarrow x[n]=-j(1+j)^nu[n]+j(1-j)^nu[n]$.

EX #3: What if there are multiple poles at the origin $z=0$? Use this trick:

$$X(z)=\frac{z^3+2z^2+3z+4}{z^2(z-1)}=\frac{z^3+2z^2+3z+4}{z^3}\frac{z}{z-1}=(1+2z^{-1}+3z^{-2}+4z^{-3})\frac{z}{z-1}$$

$\rightarrow x[n]=\{\underline{1},2,3,4\}*u[n]=u[n]+2u[n-1]+3u[n-2]+4u[n-3]$.

EX #4: $X(z)=120/[(z-1)(z-2)(z-3)(z-4)(z-5)]$.

Partial fraction expansion coefficients computation $\frac{X(z)}{z}=\frac{-1}{z}+\frac{5}{z-1}-\frac{10}{z-2}+\frac{10}{z-3}-\frac{5}{z-4}+\frac{1}{z-5}$. Computed as follows:

$A_0=(z-0)X(z)/z|_{z=0}=120/[(0-1)(0-2)(0-3)(0-4)(0-5)]=-1$.
$A_1=(z-1)X(z)/z|_{z=1}=120/[(1-0)(1-2)(1-3)(1-4)(1-5)]=5$.
$A_2=(z-2)X(z)/z|_{z=2}=120/[(2-0)(2-1)(2-3)(2-4)(2-5)]=-10$.
$A_3=(z-3)X(z)/z|_{z=3}=120/[(3-0)(3-1)(3-2)(3-4)(3-5)]=10$.
$A_4=(z-4)X(z)/z|_{z=4}=120/[(4-0)(4-1)(4-2)(4-3)(4-5)]=-5$.
$A_5=(z-5)X(z)/z|_{z=5}=120/[(5-0)(5-1)(5-2)(5-3)(5-4)]=1$.

Inverse z-xform $x[n]=-\delta[n]+5u[n]-10(2)^nu[n]+10(3)^nu[n]-5(4)^nu[n]+1(5)^nu[n]$.
Note this is an unstable signal, since it blows up as $n\rightarrow\infty$.

PARTIAL FRACTION EXPANSIONS: COMPLEX POLES

Given: $X(z) = \frac{z-1}{z^3+4z^2+8z+8}$ (Chen p. 257). **Find:** Inverse z-transform.

Poles: $z^3+4z^2+8z+8 = (z+2)(z-2e^{j2.09})(z-2e^{-j2.09})$ (from **roots**)

Form: $\frac{X(z)}{z} = \frac{z-1}{z(z+2)(z-2e^{j2.09})(z-2e^{-j2.09})} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-2e^{j2.09}} + \frac{C^*}{z-2e^{-j2.09}}$

Residues: $A = (z-0)\frac{X(z)}{z}|_{z=0} = \frac{0-1}{(0+2)(0-2e^{j2.09})(0-2e^{-j2.09})} = -\frac{1}{8}$

$B = (z+2)\frac{X(z)}{z}|_{z=-2} = \frac{-2-1}{(-2)(-2-2e^{j2.09})(-2-2e^{-j2.09})} = \frac{3}{8}$

$C = (z-2e^{j2.09})\frac{X(z)}{z}|_{z=2e^{j2.09}} = \frac{(2e^{j2.09}-1)/2e^{j2.09}}{(2e^{j2.09}+2)(2e^{j2.09}-2e^{j-2.09})} = 0.19e^{-j2.29}$

$-\frac{1}{8}\delta[n] \; +\frac{3}{8}(-2)^n u[n] + (0.19)2^n e^{j(2.09n-2.29)}u[n] + (0.19)2^n e^{-j(2.09n-2.29)}u[n]$

Using: $Ap^n + A^*(p^*)^n = 2|A||p|^n\cos(\omega_0 n+\theta)$ where $A = |A|e^{j\theta}; p = |p|e^{j\omega_0}$,

Simplify: $x(n) = -\frac{1}{8}\delta[n] + \frac{3}{8}(-2)^n u[n] + (0.38)2^n\cos(2.09n-2.29)u[n]$.

USE OF Matlab's "residue" AND "residuez" COMMANDS:

1. $X(s) = \frac{s^2+2s+1}{s^2-\frac{3}{2}s+\frac{1}{2}} = K1 + \frac{R1_1}{s-1} + \frac{R1_2}{s-\frac{1}{2}}. X(s) = 1 + \frac{\frac{7}{2}s+\frac{1}{2}}{(s-1)(s-\frac{1}{2})} \rightarrow K1 = 1.$
   $R1_1 = \frac{\frac{7}{2}s+\frac{1}{2}}{s-\frac{1}{2}}|_{s=1} = 8; R1_2 = \frac{\frac{7}{2}s+\frac{1}{2}}{s-1}|_{s=\frac{1}{2}} = -4.5.$

   >> [R1,P1,K1]=residue(B,A);[R1;P1;K1]'
   >> 8.0000 -4.5000 1.0000 0.5000 1.0000

2. $X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = K + \frac{R_1}{1-z^{-1}} + \frac{R_2}{1-\frac{1}{2}z^{-1}}$
   $(\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1}+1)|\overline{z^{-2}+2z^{-1}+1} = 2 + rem$
   $X(z) = 2 + \frac{-1+5z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \rightarrow K = 2$
   $R_1 = \frac{-1+5z^{-1}}{1-\frac{1}{2}z^{-1}}|_{z=1} = 8; R_2 = \frac{-1+5z^{-1}}{1-z^{-1}}|_{z=\frac{1}{2}} = -9.$

   >>B=[1 2 1];A=[1 -3/2 1/2];[R,P,K]=residuez(B,A);[R;P;K]'
   >>8.0000 -9.0000 1.0000 0.5000 2.0000

3. **residuez** is in the signal processing toolbox. What if it's unavailable?
   Apply **residue** to $\frac{1}{z}\frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}\frac{z^2}{z^2} = \frac{-z^2+5z+0}{z^3-\frac{3}{2}z^2+\frac{1}{2}z+0}$

   >> B=[-1 5 0];A=[1 -3/2 1/2 0];[R3,P3,K3]=residue(B,A);
   [R3;P3;K3]' >> 8.0000 -9.0000 0 1.0000 0.5000