
FREQUENCY RESPONSE OF LTI SYSTEMS

Note: $e^{j\omega_0 n} \rightarrow \overline{h[n]} \rightarrow H(e^{j\omega_0})e^{j\omega_0 n}$

where: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$ = **frequency response function.**

Huh? Single frequency into LTI system \rightarrow Same single frequency out.

Why? $y[n] = h[n] * e^{j\omega_0 n} = \sum h[i]e^{j\omega_0(n-i)} = e^{j\omega_0 n} \sum h[i]e^{-j\omega_0 i} = e^{j\omega_0 n} H(e^{j\omega_0})$.

So? $e^{+j\omega_0 n} \rightarrow \overline{h[n]} \rightarrow H(e^{+j\omega_0})e^{+j\omega_0 n}$.

and: $e^{-j\omega_0 n} \rightarrow \overline{h[n]} \rightarrow H(e^{-j\omega_0})e^{-j\omega_0 n}$.

Add: $\cos(\omega_0 n) \rightarrow \overline{h[n]} \rightarrow |H(e^{j\omega_0})| \cos(\omega_0 n + \arg[H(e^{j\omega_0})])$

since: LTI system and $H(e^{\pm j\omega_0}) = |H(e^{j\omega_0})|e^{\pm j \arg[H(e^{j\omega_0})]}$.

Gain: Amplitude increases by factor of $|H(e^{j\omega_0})|$.

Phase: Shift by $\arg[H(e^{j\omega_0})] = \tan^{-1} \frac{\text{Im}[H(e^{j\omega_0})]}{\text{Re}[H(e^{j\omega_0})]}$.

EX #1: $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = \{\dots -1, 0, 1, 0, -1 \dots\} = \cos(\frac{\pi n}{2})$.
 $H(e^{j\omega}) = 1/(1 - \frac{1}{2}e^{-j\omega}) = 1/(1 + \frac{j}{2}) = 0.89e^{-j26.6^\circ}$ at $\omega = \frac{\pi}{2}$
 $y[n] = 0.89 \cos(\frac{\pi n}{2} - 26.6^\circ) = \{\dots 0.8, 0.4, -0.8, -0.4, 0.8, 0.4 \dots\}$.

EX #2: $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = \{\dots -1, 1, -1, 1, -1 \dots\} = \cos(\pi n)$.
 $H(e^{j\omega}) = 1/(1 + \frac{1}{2}) = \frac{2}{3}$ at $\omega = \pi \rightarrow y[n] = \frac{2}{3} \cos(\pi n) = \frac{2}{3}(-1)^n$.

Why $\frac{2}{3}$? $y[n] = h[n] * x[n] = \sum (\frac{1}{2})^i (-1)^{n-i} = (-1)^n \sum (-\frac{1}{2})^i = (-1)^n \frac{1}{1 - (-\frac{1}{2})}$.

EX #3a: $h[n] = \{\frac{1}{2}, +\frac{1}{2}\} \rightarrow H(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega}) = \cos(\frac{\omega}{2})e^{-j\omega/2}$. Lowpass.

EX #3b: $h[n] = \{\frac{1}{2}, -\frac{1}{2}\} \rightarrow H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega}) = \sin(\frac{\omega}{2})e^{j(\pi-\omega)/2}$. High.

Notch: $H(e^{j\omega}) = 2 \cos(\omega) - 2 \cos(\omega_0)$. $h[n] = \{1, -2 \cos(\omega_0), 1\}$.

Why? This filters out a frequency component at frequency ω_0 .

So? Use *superposition* property of LTI systems and Fourier series:

1. Let $x[n]$ be **any** periodic signal. Let $x[n]$ have period N .
2. Then $x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$ for some constants X_k .
3. Then $X_k e^{j2\pi nk/N} \rightarrow \overline{h[n]} \rightarrow H(e^{j2\pi k/N}) X_k e^{j2\pi nk/N}$ (note $\omega_0 = \frac{2\pi k}{N}$).
4. Then $x[n] \rightarrow \overline{h[n]} \rightarrow \sum_{k=0}^{N-1} H(e^{j2\pi k/N}) X_k e^{j2\pi nk/N}$.

Huh? The k^{th} frequency component of $x[n]$ is multiplied by $H(e^{j2\pi k/N})$.

EX: $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = 1 + 2 \cos(\frac{\pi}{2}n) + 3 \cos(\pi n)$.

Then: $y[n] = \frac{1}{1 - \frac{1}{2}} 1 + |\frac{1}{1 + \frac{j}{2}}| 2 \cos(\frac{\pi}{2}n + \arg[\frac{1}{1 + \frac{j}{2}}]) + \frac{1}{1 + \frac{1}{2}} 3 \cos(\pi n)$

$\rightarrow y[n] = 2 + 1.78 \cos(\frac{\pi}{2}n - 26.6^\circ) + 2 \cos(\pi n)$.

NOTE: Plots of **several periods** of frequency response for $h[n]$.

Remember: $H(e^{j\omega})$ is periodic in ω with period 2π !

