CONTINUOUS-TIME FOURIER SERIES

Professor Andrew E. Yagle, EECS 206 Instructor, Fall 2005

Dept. of EECS, The University of Michigan, Ann Arbor, MI 48109-2122

I. Abstract

The purpose of this document is to introduce EECS 206 students to the continuous-time Fourier series, where it comes from, what it's for, and how to use it. It also shows how to display line spectra, derives Parseval's theorem, and discusses orthogonality and even and odd functions.

A. *Table of contents by sections:*

II. Fourier Series: What's it All About?

So just what is a Fourier series, and why should you care?

A. *Basic Concept of Fourier Series*

Let $x(t)$ be any real-valued periodic signal having period $= T$ seconds, so $x(t) = x(t+T)$ for *all* t.

Then $x(t)$ can be expanded as a sum of sinusoids having frequencies that are integer multiples of $\frac{1}{T}$ Hertz:

$$x(t) = c_0 + c_1\cos(\frac{2\pi}{T}t - \theta_1) + c_2\cos(\frac{4\pi}{T}t - \theta_2) + c_3\cos(\frac{6\pi}{T}t - \theta_3) + c_4\cos(\frac{8\pi}{T}t - \theta_4) + \ldots \quad (1)$$

Nomenclature for Fourier series:

- $\frac{1}{T}$ Hertz is called the **fundamental frequency**, since all others are integer multiples of this one
- c_0 is called the **DC term**, since it is constant ("DC"=Direct (constant) Current)
- $c_1\cos(\frac{2\pi}{T}t - \theta_1)$ is called the **fundamental** (musicians should be familiar with this term)
- $c_n\cos(\frac{n\pi}{T}t - \theta_n)$ is called the n^{th} *harmonic* by signal processing people, the $(n-1)^{th}$ *harmonic* by circuits people, and an *overtone* by musicians. Others number only the nonzero harmonics
- In EECS 206 you won't be asked, "What's the n^{th} harmonic?" since the answer is ambiguous

Three ways to think about Fourier series:

1. As a mathematical version of a **prism**: breaking up a signal into different frequencies, just as a prism (or diffraction grating) breaks up light into different colors (which are light at different frequencies)
2. As a mathematical depiction of **adding overtones** to a basic note to give a richer and fuller sound. It can also be used as a formula for *synthesis* of sounds and tones (see below)
3. A representation of $x(t)$ in terms of **orthogonal functions**, so the amplitudes c_n and phase shifts θ_n can be computed easily, using the explicit formulae given below, and Parseval's theorem is true.

III. Trigonometric and Exponential Series

A. Fourier Series, Have We Three

There are actually three different types of Fourier series:

$$\begin{aligned}
x(t) &= c_0 \quad + c_1\cos(\tfrac{2\pi}{T}t-\theta_1) + c_2\cos(\tfrac{4\pi}{T}t-\theta_2) + c_3\cos(\tfrac{6\pi}{T}t-\theta_3) + \ldots \\
x(t) &= a_0 \quad + a_1\cos(\tfrac{2\pi}{T}t) + a_2\cos(\tfrac{4\pi}{T}t) + a_3\cos(\tfrac{6\pi}{T}t) + a_4\cos(\tfrac{8\pi}{T}t) + \ldots \\
&\qquad + b_1\sin(\tfrac{2\pi}{T}t) + b_2\sin(\tfrac{4\pi}{T}t) + b_3\sin(\tfrac{6\pi}{T}t) + b_4\sin(\tfrac{8\pi}{T}t) + \ldots \\
x(t) &= x_0 \quad + x_1e^{j\frac{2\pi}{T}t} + x_2e^{j\frac{4\pi}{T}t} + x_3e^{j\frac{6\pi}{T}t} + x_4e^{j\frac{8\pi}{T}t} + \ldots \\
&\qquad + x_1^*e^{-j\frac{2\pi}{T}t} + x_2^*e^{-j\frac{4\pi}{T}t} + x_3^*e^{-j\frac{6\pi}{T}t} + x_4^*e^{-j\frac{8\pi}{T}t} + \ldots
\end{aligned} \tag{2}$$

The three series expand the periodic signal $x(t)$ as a sum of:

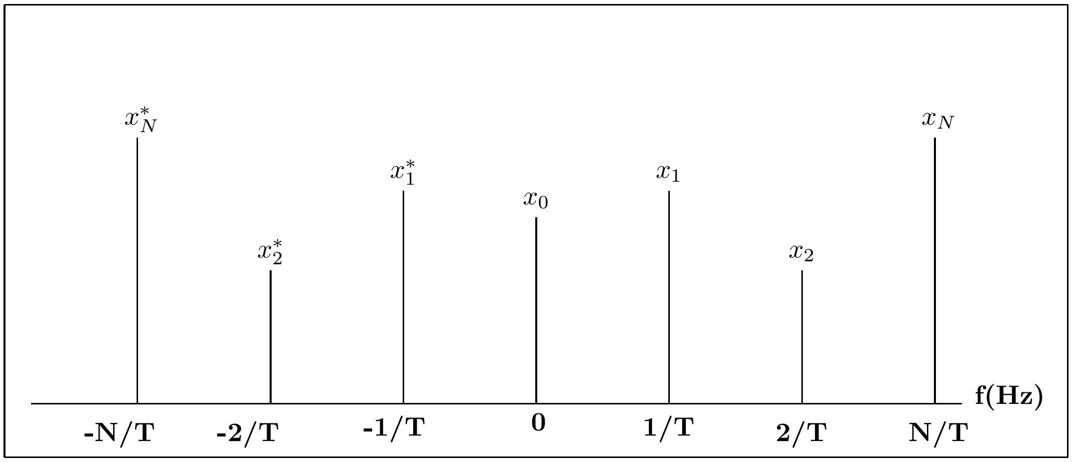
1. Phase-shifted sinusoids $c_n\cos(\frac{2\pi}{T}nt-\theta_n)$
2. Sines plus cosines $a_n\cos(\frac{2\pi}{T}nt)+b_n\sin(\frac{2\pi}{T}nt)$
3. Complex exponentials $x_ne^{j\frac{2\pi}{T}nt}$ where x_n are complex numbers

So why do we need three different Fourier series? Each has a different ease of computation:

1. This is the simplest to understand, but it requires the most work to compute it
2. This is easier to compute, and represents the even and odd parts of $x(t)$ separately
3. This is easiest to compute, analogous to the Discrete Fourier Transform (DFT), and it will prove most useful later in the course. But it is the most abstract for you right now.

B. Line Spectrum: A Visual Display of Fourier Series

Fourier series have an awful lot of numbers in them. It would be nice to have a visual depiction of them. The line spectrum of $x(t)$ given by the complex exponential form of the Fourier series above is the plot



- The *horizontal* position of each line is the frequency of the sinusoid: $\frac{n}{T}$ Hertz$=\frac{2\pi n}{T}$ $\frac{\text{RAD}}{\text{SEC}}$
- The *vertical* height of each line is half the amplitude of the sinusoid: $|x_n| = c_n/2$ for $n > 0$
- The *negative frequencies* are negatives of the positive frequencies: Height $|x_{-n}| = |x_n^*| = |x_n|$
- Each sinusoid is represented by two complex exponentials: $x_n e^{j2\pi nt/T} + x_n^* e^{-j2\pi nt/T}$
- This is also why the height of each line is only half the amplitude (except at DC: $x_0 = c_0$)

The above plot is also the line spectrum of the sum of sinusoids

$$x(t) = c_0 + c_1 \cos(\frac{2\pi}{T}t + \theta_1) + \ldots + c_N \cos(\frac{2\pi}{T}Nt + \theta_N) = x_0 + \sum_{n=1}^{N}(x_n e^{j2\pi nt/T} + x_n^* e^{-j2\pi nt/T}) \tag{3}$$

where $x_n = (c_n/2)e^{j\theta_n}$. That is, we can represent a sinusoid by *two complex exponentials*:

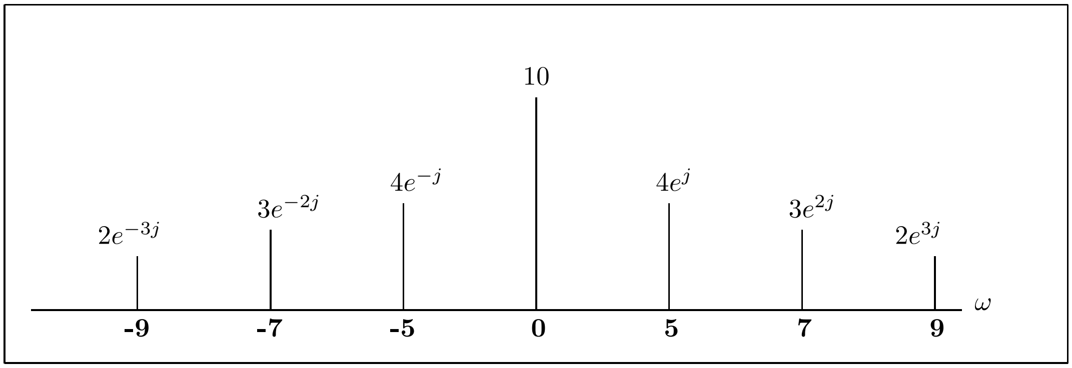
$$C\cos(\omega t + \theta) = C\frac{1}{2}\left(e^{j\omega t+\theta} + e^{-j\omega t-\theta}\right) = \left(\frac{C}{2}e^{j\theta}\right)e^{j\omega t} + \left(\frac{C}{2}e^{-j\theta}\right)e^{-j\omega t} = Xe^{j\omega t} + X^* e^{-j\omega t} \tag{4}$$

where $X = (C/2)e^{j\theta}$. This is the one tricky aspect of line spectra: they plot $|x_n|$, not $|c_n|$, at frequency n/T Hertz. You must remember to halve the amplitudes of sinusoids. A good way to remember this is that two complex exponentials combine to give one sinusoid.

The line spectrum of $x(t)$ gives an easy-to-understand visual depiction of the Fourier series of $x(t)$. It shows at a glance what frequencies in the spectrum of $x(t)$ are most significant, and which are not significant.

Example: The line spectrum of the following signal is shown on the next page:

$$\begin{aligned} x(t) = &\quad 10 + 8\cos(5t+1) + 6\cos(7t+2) + 4\cos(9t+3) \quad \text{Complex} \quad \text{exponentials}: \\ x(t) = &\quad 10 + 4e^{j1}e^{j5t} + 4e^{-j1}e^{-j5t} + 3e^{j2}e^{j7t} + 3e^{-j2}e^{-j7t} + 2e^{j3}e^{j9t} + 2e^{-j3}e^{-j9t} \end{aligned} \tag{5}$$



C. Even and Odd Functions

These will prove to be very useful concepts later, especially for computation of Fourier coefficients.

- An *even* or *symmetric* function $x(t)$ has the property $x(t) = x(-t)$
- The plot of an even function for $t < 0$ is the mirror image of the plot for $t > 0$
- Examples of even functions: $\cos(\omega_o t)$ for any ω_o; t^2; t^4; $t^6 \ldots$

- An *odd* or *anti-symmetric* function has the property $x(t) = -x(-t)$
- The plot for $t < 0$ is the plot for $t > 0$ reflected about **both** axes
- Examples of odd functions: $\sin(\omega_o t)$ for any ω_o; t; t^3; $t^5 \ldots$

An important fact is that any function $x(t)$ can be written as the sum of its *even and odd parts*:

$$x(t) = x_e(t) + x_o(t) \quad \text{where} \quad x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad \text{and} \quad x_o(t) = \frac{1}{2}[x(t) - x(-t)] \tag{6}$$

In the second Fourier series given above, we have

$$\begin{aligned} x_e(t) = \quad & a_0 + \quad a_1 \cos(\frac{2\pi}{T}t) + a_2 \cos(\frac{4\pi}{T}t) + a_3 \cos(\frac{6\pi}{T}t) + \ldots \\ x_o(t) = \quad & \qquad\;\; b_1 \sin(\frac{2\pi}{T}t) + b_2 \sin(\frac{4\pi}{T}t) + b_3 \sin(\frac{6\pi}{T}t) + \ldots \end{aligned} \tag{7}$$

Knowing this can save you a lot of work in computing coefficients of Fourier series.

Also note that ('even'=any even function; 'odd'=any odd function; 'any'=any number):

- (even)(even)=even; (odd)(odd)=even; (even)(odd)=odd.
- $\int_{-\text{any}}^{+\text{any}} \text{odd}(t)dt = 0$ and $\int_{-\text{any}}^{+\text{any}} \text{even}(t)dt = 2\int_0^{+\text{any}} \text{even}(t)dt$.

IV. Examples of Fourier Series Expansions

A. *Finite Fourier Series*

Rarely, the Fourier series of $x(t)$ has only a finite number of terms.

Let a simple sinusoid $x(t) = A\cos(\omega t)$ be passed through a *square-law* device. Thermocouples and some radiation detectors are square-law devices whose output $y(t)$ is the square of the input $x(t)$. Using trig,

$$y(t) = x(t)^2 = A^2\cos^2(\omega t) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega t) \tag{8}$$

which is indeed a Fourier series with

$$c_0 = a_0 = \frac{A^2}{2}; \quad c_2 = a_2 = \frac{A^2}{2} \tag{9}$$

with all other a_n, b_n, c_n, θ_n equal to zero.

What good is this result? Suppose that $x(t)$ is a red laser beam (like that used in a laser pointer). $x(t)$ has wavelength (i.e., its period in space, rather than time) of 6328 Angstrom, so it is a sinusoid with frequency

$$\frac{3\times 10^8\,\frac{\text{METER}}{\text{SECOND}}}{6328\times 10^{-10}\,\text{METER}} = 4.74\times 10^{14}\,\text{HERTZ} = 474\,\text{TERAHERTZ} \tag{10}$$

If the red laser is shown through a nonlinear crystal that squares the electromagnetic wave, the output will have a DC component and a component at double the frequency, i.e., 3164 Angstrom, which is ultra-violet light. Since the energy of each photon doubles (remember $E = h\nu$?), there are only half as many photons.

While this is a simple result, what if the nonlinear crystal does something more complex than squaring? Then the Fourier series expansion of the output function $y(t)$ literally gives the spectrum of the output!

B. *Square Waves and Triangle Waves*

These are the typical examples of Fourier series, and they do illustrate how the Fourier series converges to $x(t)$ (you don't really believe it until you see it). First, let $x(t)$ be the zero-mean *square wave*

$$x(t) = \begin{cases} \pi/4 & \text{for } (2n\pi) < t < (2n+1)\pi \quad \text{Square wave with period} \\ -\pi/4 & \text{for } (2n-1)\pi < t < (2n\pi) \quad T = 2\pi \rightarrow \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \end{cases} \tag{11}$$

for all integers n. We will show below that the Fourier series expansion of $x(t)$ is:

$$x(t) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + \frac{1}{9}\sin(9t) + \dots \tag{12}$$

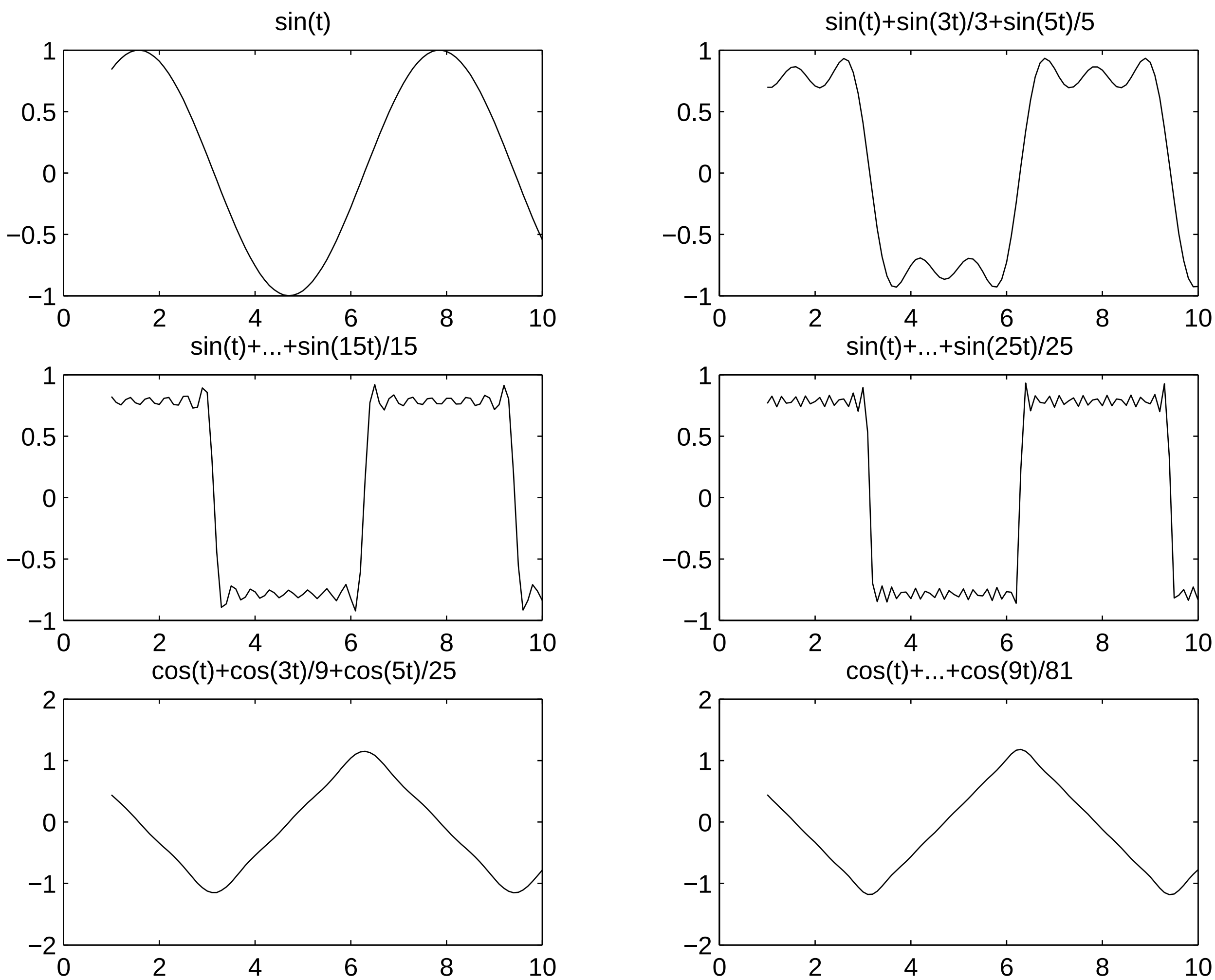
Note that:

- $x(t)$ is an odd function $\rightarrow b_n = 0$
- The DC term is zero since $x(t)$ is zero-mean

- The fundamental frequency is $1\frac{\text{RAD}}{\text{SEC}} = \frac{1}{2\pi}$ Hertz
- The harmonics are at frequencies $\frac{2n+1}{2\pi}$ Hertz, for integers n
- Even-numbered harmonics are zero (according to signal processing people)

The figure below plots some partial sums of this series. Note that:

- The Fourier series is clearly converging to the square wave
- At the discontinuities at $t = n\pi$, convergence is to the midpoint
- The ripples get smaller and faster as we include more terms
- The jumps at $t = n\pi$ get faster as we include more terms
- The overshoot at the end of each jump does *not* disappear as we add more terms.
- This is called *Gibbs's phenomenon*; take EECS 306 for details about it.



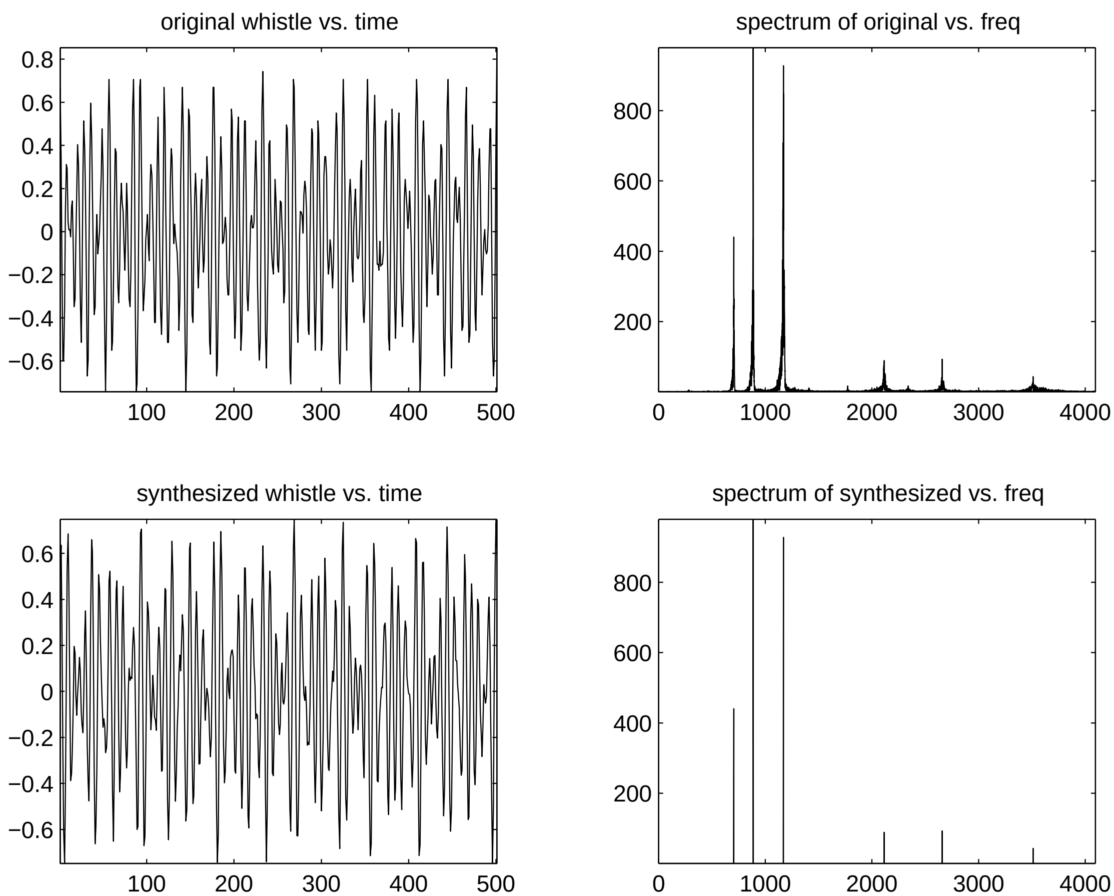
The above figure also plots some partial sums of the Fourier series for a *triangle wave*. Note that:

- This Fourier series is converging much faster than the square wave one
- The Fourier series coefficients now drop off as $1/k^2$, not $1/k$
- There are no overshoots, since there are no discontinuities

C. Real-World Signal: Train Whistle

Here $x(t)$ is a recorded train whistle. Its Fourier series would take too long to write out, but the line spectrum computed using the DFT (see my DFT notes for details) is depicted. Note that:

- The train whistle has a lot of sinusoids in it, but most have very small amplitudes
- In fact, a half-dozen sinusoids do a decent job of representing the train whistle
- So if you want to *compress* or *synthesize* the train whistle sound, you can either:
  - Store 12880 samples $x(n/8192)$ of the continuous-time signal $x(t)$, **OR:**
  - Store 6 amplitudes, phases, and frequencies. You make the call!
  - This is the basic idea behind compression (MP3 for audio; JPEG for images)
  - In problem set #6 you will be exploring this possibility yourself



The approximate train whistle signal using the half-dozen sinusoids is:

$$\begin{aligned} x(t) &= 440.5\cos(2\pi 705t - 1.70) + 979.0\cos(2\pi 886t + 2.93) + 928.0\cos(2\pi 1170t + 1.35) \\ & \quad 88.36\cos(2\pi 2115t + 1.41) + 92.92\cos(2\pi 2658t + 0.84) + 42.81\cos(2\pi 3511t + 3.03) \end{aligned} \tag{13}$$

Where did I get those numbers (6 amplitudes, phases, and frequencies)?

1. I used Matlab's `load train.mat;plot(y)` to obtain the train whistle signal (which is included in Matlab) sampled at `Fs`=8192 Hz⇔8192 samples per second⇔one sample every 1/8192 seconds
2. `length y`=12880 samples, so the duration of the train whistle signal is $\frac{12880}{8192}$=1.57 seconds
3. `F=fft(y);` approximates the integral formula for computing Fourier series coefficients $\{x_n, 0 \le n \le \frac{12880}{2}\}$ (actually, it's *not* an approximation for bandlimited signals–see my DFT notes)
4. `plot(abs(F))` plots the line spectrum; `[abs(F)/2 angle(F)]` gives all of the amplitudes and phases
5. `F(K)` is the component at frequency $f = (K-1)\frac{8192}{12880}$ Hertz, since Matlab indexing starts at 1 not 0
6. The plots vs. time are actually the samples $x(n/8192)$ for $11000 \le n \le 11500$ (to avoid end effects)
7. Try listening to the original and synthesized train whistles using `soundsc(y)` and `soundsc(X)`

Examining the six frequencies above is instructive. Note that the highest three frequencies are each triple the lowest three frequencies, strongly suggesting that they are harmonics of three fundamental frequencies. The three lowest frequencies are related (roughly) by 5:3 and 5:4 ratios; likely this is not a coincidence.

Musical *chords* are pairs or triplets of sinusoids at frequencies related by simple ratios like 5:3 or 5:4 etc.

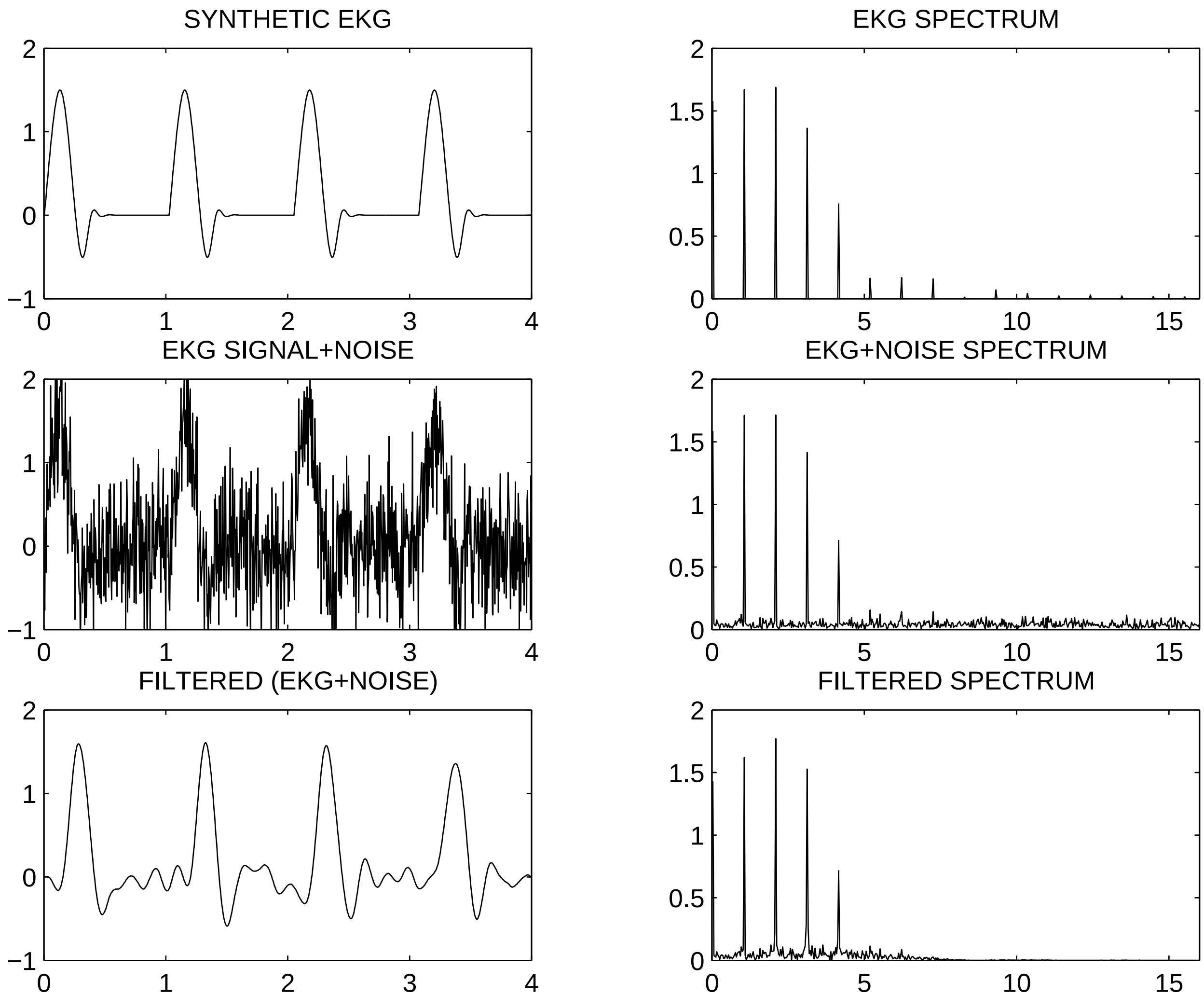
D. Real-World Signal: Electrocardiogram (EKG)

This is actually a synthetic EKG, but this is (roughly) what they look like. EKGs are good signals for understanding Fourier series, since they are naturally periodic with few harmonics. See the next page:

- The *fundamental frequency* is 60 $\frac{\text{BEATS}}{\text{MINUTE}}$=1 Hertz
- The Fourier series coefficients drop off rapidly
- The EKG signal has almost no components above 25 Hertz

The figure shows how we can use this last fact to *filter out* noise. Since everything above 25 Hertz can be eliminated without affecting the EKG signal, we filter the noisy EKG signal and eliminate all components above 25 Hertz. That gets rid of most of the noise. We can't do anything about the noise components below 25 Hertz; we can't eliminate them without also affecting the EKG signal (they are added together).

This shows you what *filtering* means. But how do we *do* it? Stay tuned to EECS 206...



V. Computing Fourier Series Coefficients

I put this off as long as possible; here's where the math begins. But before you swear off signal processing and go into circuits, note that **you won't have to compute messy stuff like this, even in EECS 206**. The DFT (Discrete Fourier Transform) will save you! But you do need to know from what you are saved...

A. *Formulae for Fourier Coefficients*

The good news is that there are explicit formulae for computing all of the various Fourier series coefficients

$$\begin{aligned}
x_k &= \tfrac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt && \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3 \ldots \\
a_0 &= \tfrac{1}{T}\int_0^T x(t)dt = M(x) && \text{for} \quad k = 0 \\
a_k &= \tfrac{2}{T}\int_0^T x(t)\cos(\tfrac{2\pi}{T}kt)dt && \text{for} \quad k = 1, 2, 3 \ldots \\
b_k &= \tfrac{2}{T}\int_0^T x(t)\sin(\tfrac{2\pi}{T}kt)dt && \text{for} \quad k = 1, 2, 3 \ldots \\
c_k &= \sqrt{a_k^2 + b_k^2} \quad \text{and} && \theta_k = \tan^{-1}(b_k/a_k)(+\pi?)
\end{aligned} \tag{14}$$

B. Properties of Fourier Coefficients

- $x_k = (a_k - jb_k)/2 = (c_k/2)e^{j\theta_k}$, provided $k \neq 0$
- $a_k = x_k + x_{-k} = 2\,Re[x_k] = \frac{2}{T}\int x(t)\cos(2\pi kt/T)dt$
- Watch the signs in the following equations:
- $b_k = j(x_k - x_{-k}) = -2\,Im[x_k] = \frac{2}{T}\int x(t)\sin(2\pi kt/T)dt$
- For even and odd parts we have explicitly:
- $x_o(t) = \sum_{k=1}^{\infty} b_k \sin(2\pi kt/T); \quad x_e(t) = \sum_{k=0}^{\infty} a_k \cos(2\pi kt/T)$

Note that:

- x_k are the easiest to compute, and even easier if $x(t)$ is real:
- If $x(t)$ is real, then $x_{-k} = x_k^*$; this is called *conjugate symmetry*
- If $x(t)$ is even, then $b_k = 0$; this cuts our workload in half
- If $x(t)$ is odd, then $a_k = 0$; this cuts our workload in half
- $a_0 = M(x)$ is computed differently from $\{a_k, k \neq 0\}$ (have to remember this)

The bad news is that these formulae can be a real pain. Don't believe me? Read on...

VI. Examples of Computing Fourier Series

A. Square Wave

Let's compute the Fourier series coefficients of the square wave shown above. That looks pretty simple. But watch what happens. A single period of $x(t)$ is:

$$x(t) = \begin{cases} +\pi/4 & \text{for } 0 < t < \pi \quad \text{and periodic with period} = \mathrm{T} = 2\pi. \\ -\pi/4 & \text{for } \pi < t < 2\pi \quad \textbf{Goal}: \text{Compute its Fourier series.} \end{cases} \tag{15}$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos(\frac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-\frac{\pi}{4})\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}(\frac{\pi}{4})\cos(nt)dt \\ &= -\frac{1}{4n}\sin(nt)|_{-\pi}^{0} + \frac{1}{4n}\sin(nt)|_{0}^{\pi} = 0 - 0 + 0 - 0 = 0 \quad \text{since} \quad \sin(n\pi) = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin(\frac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-\frac{\pi}{4})\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}(\frac{\pi}{4})\sin(nt)dt \\ &= \frac{1}{4n}\cos(nt)|_{-\pi}^{0} - \frac{1}{4n}\cos(nt)|_{0}^{\pi} = \frac{1}{4n}(1 - \cos(-\pi n)) - \frac{1}{4n}(\cos(\pi n) - 1) \\ &= \frac{1}{n}\frac{1}{2}(1 - \cos(\pi n)) = \begin{cases} 1/n & \text{for n odd;} \\ 0 & \text{for n even} \end{cases} \quad \text{since} \quad \cos(\pi n) = (-1)^n \end{aligned} \tag{17}$$

This is awful! Isn't there any way to simplify this computation? Yes:even and odd functions! $x(t)$ is an **odd** function (reflect it about both [not each] axes). Then, instead of four integrals, we need compute only one:

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos(\frac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(odd)(even)dt = 0 \\
b_n &= \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin(\frac{2\pi}{T}nt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(odd)(odd)dt = \frac{2}{\pi}\int_{0}^{\pi}(even)dt \\
\text{which in this case is } b_n &= \frac{2}{\pi}\int_{0}^{\pi}(\frac{\pi}{4})\sin(nt)dt = \frac{1}{n}\frac{1}{2}(1-\cos(\pi n))
\end{aligned} \tag{18}
$$

You have to admit that's a lot easier! Still easier is to use complex exponentials:

$$
\begin{aligned}
x_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)e^{-j\frac{2\pi}{T}nt}dt &= \frac{1}{2\pi}\int_{-\pi}^{0}(-\frac{\pi}{4})e^{-jnt}dt + \frac{1}{2\pi}\int_{0}^{\pi}(\frac{\pi}{4})e^{-jnt}dt \\
&= \frac{1}{8jn}e^{-jnt}|_{-\pi}^{0} - \frac{1}{8jn}e^{-jnt}|_{0}^{\pi} = \frac{1}{4jn}(1-e^{-j\pi n}) = \frac{-j}{2n}, \quad n\,\text{odd}
\end{aligned} \tag{19}
$$

Plugging in to the complex Fourier series expansion gives

$$
\begin{aligned}
x(t) &= \frac{1}{2j}e^{+jt} + \frac{1}{6j}e^{+j3t} + \frac{1}{10j}e^{+j5t} + \frac{1}{14j}e^{+j7t} + \frac{1}{18j}e^{+j9t} + \ldots \\
&\quad -\frac{1}{2j}e^{-jt} - \frac{1}{6j}e^{-j3t} - \frac{1}{10j}e^{-j5t} - \frac{1}{14j}e^{-j7t} - \frac{1}{18j}e^{-j9t} - \ldots \\
x(t) &= \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + \frac{1}{9}\sin(9t) + \ldots
\end{aligned} \tag{20}
$$

which shows that the formulae all do give the same Fourier series expansion (sum vertically).

VII. Derivation of Computing Fourier Series

The derivations of the above formulae are actually very simple, if you don't get lost in the math. They rely heavily on the *orthogonality* of the complex exponentials and of sinusoids at integer multiples of a fundamental frequency $1/T$ Hertz.

A. Orthogonality of Basis Functions

Specifically, we have the four orthogonality equations

$$
\begin{aligned}
0 &= \int_0^T e^{j\frac{2\pi}{T}mt}e^{-j\frac{2\pi}{T}nt}dt && \text{unless} \quad m=n \\
0 &= \int_0^T \cos(\tfrac{2\pi}{T}mt)\cos(\tfrac{2\pi}{T}nt)dt && \text{unless} \quad m=n \\
0 &= \int_0^T \sin(\tfrac{2\pi}{T}mt)\sin(\tfrac{2\pi}{T}nt)dt && \text{unless} \quad m=n \\
0 &= \int_0^T \sin(\tfrac{2\pi}{T}mt)\cos(\tfrac{2\pi}{T}nt)dt && \text{even if} \quad m=n
\end{aligned} \tag{21}
$$

The first equation is

$$\int_0^T e^{j2\pi(m-n)t/T}dt = \int_0^T \cos(2\pi(m-n)t/T)dt + j\int_0^T \sin(2\pi(m-n)t/T)dt = 0 + j0 = 0 \tag{22}$$

since the mean value of a sinusoid with nonzero frequency over a period is zero. The remaining three equations can be derived from the first by multiplying two of the following:

$$\begin{aligned} \cos(2\pi mt/T) = & \quad [e^{j2\pi mt/T} + e^{-j2\pi mt/T}]/2 \\ \sin(2\pi mt/T) = & \quad [e^{j2\pi mt/T} - e^{-j2\pi mt/T}]/(2j) \end{aligned} \tag{23}$$

with m replaced with n as needed, integrating from 0 to T, and using the first equation.

Another way of deriving the remaining three equations is to use trig identities (ugh)

$$\begin{aligned} 2\cos(2\pi mt/T)\cos(2\pi nt/T) &= \cos(2\pi(m-n)t/T) + \cos(2\pi(m+n)t/T) \\ 2\sin(2\pi mt/T)\sin(2\pi nt/T) &= \cos(2\pi(m-n)t/T) - \cos(2\pi(m+n)t/T) \\ 2\cos(2\pi mt/T)\sin(2\pi nt/T) &= \sin(2\pi(m+n)t/T) - \sin(2\pi(m-n)t/T) \end{aligned} \tag{24}$$

Integrating these from 0 to T and using the fact that the mean value of a sinusoid over a period is zero, *unless the sinusoid has zero frequency*, immediately yields the orthogonality equations.

B. Derivation of Formulae

These follow immediately from the above orthogonality equations and the original Fourier series. For example, take the complex exponential form of the Fourier series, multiply by $e^{-j\frac{2\pi}{T}kt}$ and integrate from 0 to T. By the first orthogonality equation, all terms in the Fourier series except the k^{th} term are zero:

$$\int_0^T x(t)e^{-j\frac{2\pi}{T}kt} = 0 + 0 + \ldots + 0 + Tx_k + 0 + \ldots \tag{25}$$

This immediately gives us the formula for x_k. Similar arguments may be used to obtain the formulae for a_k and b_k. Note how important the orthogonality of the complex exponentials and of the sinusoids is here; without it, there would be no clear way to obtain the Fourier coefficients.

The formulae for $\{c_k, \theta_k\}$ are derived using phasors:

$$a\cos(\omega t) + b\sin(\omega t) \to a - jb = \sqrt{a^2+b^2}e^{-j\tan^{-1}(b/a)} \to \sqrt{a^2+b^2}\cos\left(\omega t - \tan^{-1}(b/a)\right) \tag{26}$$

with the usual caveat that π must be added or subtracted (same result) from the phase if $a < 0$.

VIII. Parseval's Theorem and Orthogonality

A. Average Power

Parseval's theorem states that we can compute *average power* in either the time or frequency domains:

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |x_k|^2 = a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)/2 \tag{27}$$

For the square wave example above, Parseval's theorem states that

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} |\pm\frac{\pi}{4}|^2 dt = \frac{\pi^2}{16} = \frac{1}{2}\sum_{k\,\text{odd}}^{\infty}(\frac{1}{k})^2 = \frac{1}{2}(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots) = \frac{\pi^2}{16} \tag{28}$$

This infinite series does in fact sum to $\pi^2/16$; try computing its partial sums numerically.

Recall that the *rms* value of a sinusoid is its amplitude/$\sqrt{2}$, and the average power of a sinusoid=(rms)^2, unless its frequency is zero. Then we have:

- Average power of the constant a_0 is a_0^2
- Average power of $a_k\cos(2\pi kt/T)$ is $a_k^2/2$
- Average power of $b_k\sin(2\pi kt/T)$ is $b_k^2/2$
- Average power of $x_k e^{j2\pi kt/T}$ is $|x_k|^2$

B. Orthogonality: Power of Sum=Sum of Powers

Also note that the average power of the sum of two *orthogonal* signals is the sum of their average powers. To derive this, let $x(t)$ and $y(t)$ be orthogonal signals, which means that their *correlation* is zero:

$$\int x(t)y(t)^* dt = C(x,y) = M(xy^*) = 0 \quad \text{and} \quad M(yx^*) = (M(xy^*))^* = 0 \tag{29}$$

Note that the second function is complex conjugated. Then we have

$$MS[x+y] = M[|x+y|^2] = M[(x+y)(x+y)^*] = M[xx^*]+M[yy^*]+M[xy^*]+M[yx^*] = MS[x]+MS[y] \tag{30}$$

Hence the average power of the sum of orthogonal signals is the sum of their average powers.

This leads directly to *Parseval's theorem*: we can compute the average power of $x(t)$ by summing the average powers of each term of its Fourier series, since all of these terms are all orthogonal to each other.

Also note that the even $x_e(t)$ and odd $x_o(t)$ parts of $x(t)$ are orthogonal, since the cosines used to represent $x_e(t)$ are orthogonal to the sines used to represent $x_o(t)$. So their average powers also add.

C. More Mathematical Details on Fourier Series

We won't try to prove existence of Fourier series in EECS 206; you need a high-level math course for it.

At a meeting of the Paris Academy in 1807, Jean-Baptiste Fourier claimed any periodic function could be expanded in sinusoids. Joseph Lagrange stood up and said he was wrong. This led to a big argument (well, they didn't have talk shows then). After a long time, the issue was finally settled as follows:

A periodic function can be expanded in sinusoids if the following *(Peter) Dirichlet Conditions* are fulfilled (these are sufficient, not necessary): Over each period (any interval of length T):

- $x(t)$ has a **finite number** of discontinuities, maxima, and minima
- $x(t)$ is **absolutely integrable**: $\int_{-T/2}^{T/2} |x(t)|dt < \infty$
- **Then:** $\underset{N\to\infty}{\text{LIM}} |x(t) - \sum_{k=-N}^{N} x_k e^{j2\pi kt/T}| = 0$ for all t
- This is called *pointwise convergence*: the series converges to $x(t)$ where $x(t)$ is defined
- At discontinuities of $x(t)$ at t_i, convergence is to $\frac{1}{2}(x(t_i^+) + x(t_i^-))$

These conditions hold for any real-world signal, but mathematicians get their kicks from finding weird functions for which conditions like this do not hold. For example:

- $x(t) = 1/(1-t)$ $\rightarrow$no Fourier series, since it is not absolutely integrable.
- $x(t) = \sin(1/t)$ $\rightarrow$no Fourier series, since ∞ maxima and minima.

We also have this weaker condition, which states that mean-square-error converges to zero:

- Let $x(t)$ have finite energy in one period: $\int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$. Then
- $\underset{N\to\infty}{\text{LIM}} \int_{-T/2}^{T/2} |x(t) - \sum_{k=-N}^{N} x_k e^{j2\pi kt/T}|^2 dt = 0.$
- This is called *mean-square convergence*; it is weaker than pointwise convergence

What about Gibbs's phenomenon? No matter how large N gets, the height of the overshoot stays constant at 8.9% of the size of the jump at the discontinuity. But the width (in t) of the overshoot goes to zero. So the *mean square* error goes to zero; this is mean-square convergence. And pointwise convergence to $x(t)$ is only required *where $x(t)$ is defined*. The Fourier series converges to the midpoint of the jump at the discontinuity, but since $x(t)$ isn't defined there, there is no error.

Stuff like this is why math analysis courses like Math 451 exist. And if you think all this is something, wait until you find out about convergence of sequences of random variables, which is even messier.