FOURIER SERIES OF PERIODIC SIGNALS

**THEOREM:** Let \( x(t) \) be a bounded periodic signal with period \( T \).
Then \( x(t) \) can be expanded as a weighted sum of sinusoids with angular frequencies that are integer multiples of \( \omega_0 = \frac{2\pi}{T} \):

\[
x(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots
+n \cos(\omega_0 t) + a_2 \sin(2\omega_0 t) + a_3 \sin(3\omega_0 t) + \ldots
\]

This is the *trigonometric Fourier series expansion* of \( x(t) \).

Or, we can use only cosines with phase shifts:

\[
x(t) = a_0 + c_1 \cos(\omega_0 t - \phi_1) + c_2 \cos(2\omega_0 t - \phi_2) + c_3 \cos(3\omega_0 t - \phi_3) + \ldots
\]

**PROOF:** Take a high-level math course to see this done properly.

**NOTE:** A Fourier series is a mathematical version of a *prism*.

**COMPUTATION OF FOURIER SERIES COEFFICIENTS:**

**THEOREM:** Coefficients \( a_n, b_n, c_n \) and \( \phi_n \) can be computed using:

\[
a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\omega_0 t) dt
\]

\[
b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\omega_0 t) dt
\]

For \( n = 0 \) we have: \( a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt = \text{average value of } x(t) \).

For cosines with phase shifts: \( c_n = \sqrt{a_n^2 + b_n^2}; \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right), n \neq 0 \)
using \( a_n \cos(\omega t) + b_n \sin(\omega t) = \sqrt{a_n^2 + b_n^2} \cos(\omega t - \tan^{-1}\left(\frac{b_n}{a_n}\right)) \) (\( a_n > 0 \)).

Derive using phasors: \( a_n + b_n e^{-j\pi/2} = a_n - jb_n = \sqrt{a_n^2 + b_n^2} e^{-j\tan^{-1}\left(\frac{b_n}{a_n}\right)} \).

**EXAMPLE OF FOURIER SERIES DECOMPOSITION:**

\[
x(t) = \begin{cases} \pi/4 & \text{for } (2k\pi) < t < (2k + 1)\pi \quad \text{Square wave with period} \\ -\pi/4 & \text{for } (2k - 1)\pi < t < (2k\pi) \quad T = 2\pi \rightarrow \omega_0 = \frac{2\pi}{2\pi} = 1 \end{cases}
\]

\[
a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} x(t) \cos(n\omega_0 t) dt = 0 \text{ by inspection: Set } t_0 = -\frac{T}{2} = -\pi.
\]

\[
b_n = \frac{2}{2\pi} \int_{0}^{2\pi} x(t) \sin(nt) dt = \frac{1}{\pi} [\int_{0}^{\pi} \left(\frac{\pi}{4}\right) \sin(nt) dt + \int_{\pi}^{2\pi} \left(-\frac{\pi}{4}\right) \sin(nt) dt]
\]

\[= \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases} \rightarrow x(t) = \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \ldots
\]

**PARSEVAL’S THEORM FOR THIS EXAMPLE**

Average \( \frac{1}{T} \int_{0}^{T} |x(t)|^2 dt = \frac{1}{2\pi} \int_{0}^{2\pi} |x(t)|^2 dt = \frac{\pi^2}{16} \). This agrees with:

**Power:**

\[
a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{2} \sum_{k=\text{odd}}^{\infty} (\frac{1}{k})^2 = \frac{1}{2} (1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots) = \frac{\pi^2}{16}.
\]
PROOF OF THEOREM: First we need the following lemma:

**LEMMA:** The sine and cosine functions are orthogonal functions:
\[
\int_{t_0}^{t_0+T} \cos(i \omega_0 t) \cos(j \omega_0 t) dt = \int_{t_0}^{t_0+T} \sin(i \omega_0 t) \sin(j \omega_0 t) dt = \begin{cases} T/2, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}
\]
\[
\int_{t_0}^{t_0+T} \cos(i \omega_0 t) \sin(j \omega_0 t) dt = 0 \quad \text{even if } i = j.
\]
These assume \( i, j > 0 \).

**PROOF OF LEMMA:** Adding and subtracting the cosine addition formula \( \cos(x \pm y) = \cos(x) \cos(y) \mp \sin(x) \sin(y) \) gives the formulae
\[
2 \cos(x) \cos(y) = \cos(x - y) + \cos(x + y)
\]
\[
2 \sin(x) \sin(y) = \cos(x - y) - \cos(x + y).
\]

Setting \( x = i \omega_0 t \) and \( y = j \omega_0 t \) and \( \int_{t_0}^{t_0+T} dt \) gives
\[
2 \int_{t_0}^{t_0+T} \cos(i \omega_0 t) \cos(j \omega_0 t) dt = \int_{t_0}^{t_0+T} [\cos((i + j) \omega_0 t) + \cos((i - j) \omega_0 t)] dt.
\]
Since the integral of a sinusoid with nonzero frequency over an integer number \( i \pm j \) of periods is zero, the first part of the lemma follows. The other two parts follow similarly. QED.

**PROOF:** Multiply the Fourier series by \( \cos(n \omega_0 t) \) and \( \int_{t_0}^{t_0+T} dt \):
\[
\int_{t_0}^{t_0+T} x(t) \cos(n \omega_0 t) dt = \int_{t_0}^{t_0+T} a_0 \cos(n \omega_0 t) dt \\
+ \int_{t_0}^{t_0+T} a_1 \cos(n \omega_0 t) \cos(\omega_0 t) dt + \int_{t_0}^{t_0+T} a_2 \cos(n \omega_0 t) \cos(2 \omega_0 t) dt + \ldots \\
+ \int_{t_0}^{t_0+T} b_1 \cos(n \omega_0 t) \sin(\omega_0 t) dt + \int_{t_0}^{t_0+T} b_2 \cos(n \omega_0 t) \sin(2 \omega_0 t) dt + \ldots \\
= 0 + 0 + \ldots + 0 + a_n \frac{T}{2} + 0 + \ldots \text{ from which the } a_n \text{ formula follows.}
\]
The formulae for \( a_0 \) and \( b_n \) follow similarly. QED.

**COMMENTS:**

1. \( t_0 \) is arbitrary; all integrals are over one period.
2. \( \omega_0 = \frac{2\pi}{T} \) is angular frequency in \( \text{Radian} \over \text{Second} \); this is \( \frac{1}{T} \) Hertz.
3. The sinusoid at frequency \( \omega_0 \) is called the fundamental; the sinusoid at frequency \( n \omega_0 \) is the \( n^{\text{th}} \) harmonic. Harmonics are also called overtones (not used much anymore). Some say the \( n^{\text{th}} \) harmonic is at frequency \( (n + 1) \omega_0 \).
4. The more terms (i.e., harmonics) we keep in the Fourier series, the better the approximation the truncated series is to \( x(t) \).
5. If \( x(t) \) has a discontinuity at \( t = t_1 \), the Fourier series converges to \( \frac{1}{2}(x(t_1^-) + x(t_1^+)) \), where \( x(t^-) \) and \( x(t^+) \) are the values of \( x(t) \) on either side of the discontinuity.