FOURIER SERIES OF PERIODIC SIGNALS

THEOREM: Let $x(t)$ be a bounded periodic signal with period T. Then $x(t)$ can be expanded as a weighted sum of sinusoids with angular frequencies that are integer multiples of $\omega_0 = \frac{2\pi}{T}$:

$$x(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + a_3\cos(3\omega_0 t) + \ldots$$
$$+ b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + b_3\sin(3\omega_0 t) + \ldots$$

This is the *trigonometric Fourier series expansion* of $x(t)$.

Or, we can use only cosines with phase shifts:

$$x(t) = a_0 + c_1\cos(\omega_0 t - \phi_1) + c_2\cos(2\omega_0 t - \phi_2) + c_3\cos(3\omega_0 t - \phi_3) + \ldots$$

PROOF: Take a high-level math course to see this done properly.
NOTE: A Fourier series is a mathematical version of a **prism**.

COMPUTATION OF FOURIER SERIES COEFFICIENTS:

THEOREM: Coefficients a_n, b_n, c_n and ϕ_n can be computed using:

$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)dt$ and $b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)dt$.

For $n = 0$ we have: $a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$=average value of $x(t)$.

For cosines with phase shifts: $c_n = \sqrt{a_n^2 + b_n^2}; \phi_n = \tan^{-1}(\frac{b_n}{a_n}), n \neq 0$

using $a_n\cos(\omega t) + b_n\sin(\omega t) = \sqrt{a_n^2 + b_n^2}\cos(\omega t - \tan^{-1}(\frac{b_n}{a_n}))$ $(a_n > 0)$.

Derive using phasors: $a_n + b_n e^{-j\pi/2} = a_n - jb_n = \sqrt{a_n^2 + b_n^2}e^{-j\tan^{-1}(\frac{b_n}{a_n})}$.

EXAMPLE OF FOURIER SERIES DECOMPOSITION:

$x(t) = \begin{cases} \pi/4 & \text{for } (2k\pi) < t < (2k+1)\pi \\ -\pi/4 & \text{for } (2k-1)\pi < t < (2k\pi) \end{cases}$ Square wave with period $T = 2\pi \rightarrow \omega_o = \frac{2\pi}{2\pi} = 1$

$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x(t)\cos(n\omega_0 t)dt = 0$ by inspection: Set $t_0 = -\frac{T}{2} = -\pi$.

$b_n = \frac{2}{2\pi}\int_0^{2\pi} x(t)\sin(nt)dt = \frac{1}{\pi}[\int_0^{\pi}(\frac{\pi}{4})\sin(nt)dt + \int_{\pi}^{2\pi}(-\frac{\pi}{4})\sin(nt)dt]$

$= \begin{cases} \frac{1}{n}, & \text{if n is odd;} \\ 0, & \text{if n is even.} \end{cases} \rightarrow x(t) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots$

PARSEVAL'S THEORM FOR THIS EXAMPLE

Average $\frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} |\pm\frac{\pi}{4}|^2 dt = \frac{\pi^2}{16}$. This agrees with:

Power: $a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{2}\sum_{k \text{ odd}}^{\infty}(\frac{1}{k})^2 = \frac{1}{2}(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots) = \frac{\pi^2}{16}$.

PROOF OF THEOREM: First we need the following lemma:

LEMMA: The sine and cosine functions are *orthogonal* functions:

$$\int_{t_0}^{t_0+T}\cos(i\omega_0 t)\cos(j\omega_0 t)dt = \int_{t_0}^{t_0+T}\sin(i\omega_0 t)\sin(j\omega_0 t)dt = \begin{cases} T/2, & \text{if } i=j; \\ 0, & \text{if } i\neq j. \end{cases}$$

$\int_{t_0}^{t_0+T}\cos(i\omega_0 t)\sin(j\omega_0 t)dt = 0$ (even if $i=j$). These assume $i,j>0$.

PROOF OF LEMMA: Adding and subtracting the cosine addition formula $\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$ gives the formulae

$$2\cos(x)\cos(y) = \cos(x-y)+\cos(x+y)$$
$$2\sin(x)\sin(y) = \cos(x-y)-\cos(x+y).$$

Setting $x=i\omega_0 t$ and $y=j\omega_0 t$ and $\int_{t_0}^{t_0+T} dt$ gives

$$2\int_{t_0}^{t_0+T}\cos(i\omega_0 t)\cos(j\omega_0 t)dt = \int_{t_0}^{t_0+T}[\cos((i+j)\omega_0 t)+\cos((i-j)\omega_0 t)]dt.$$

Since the integral of a sinusoid with nonzero frequency over an integer number $i\pm j$ of periods is zero, the first part of the lemma follows. The other two parts follow similarly. QED.

PROOF: Multiply the Fourier series by $\cos(n\omega_0 t)$ and $\int_{t_0}^{t_0+T} dt$:

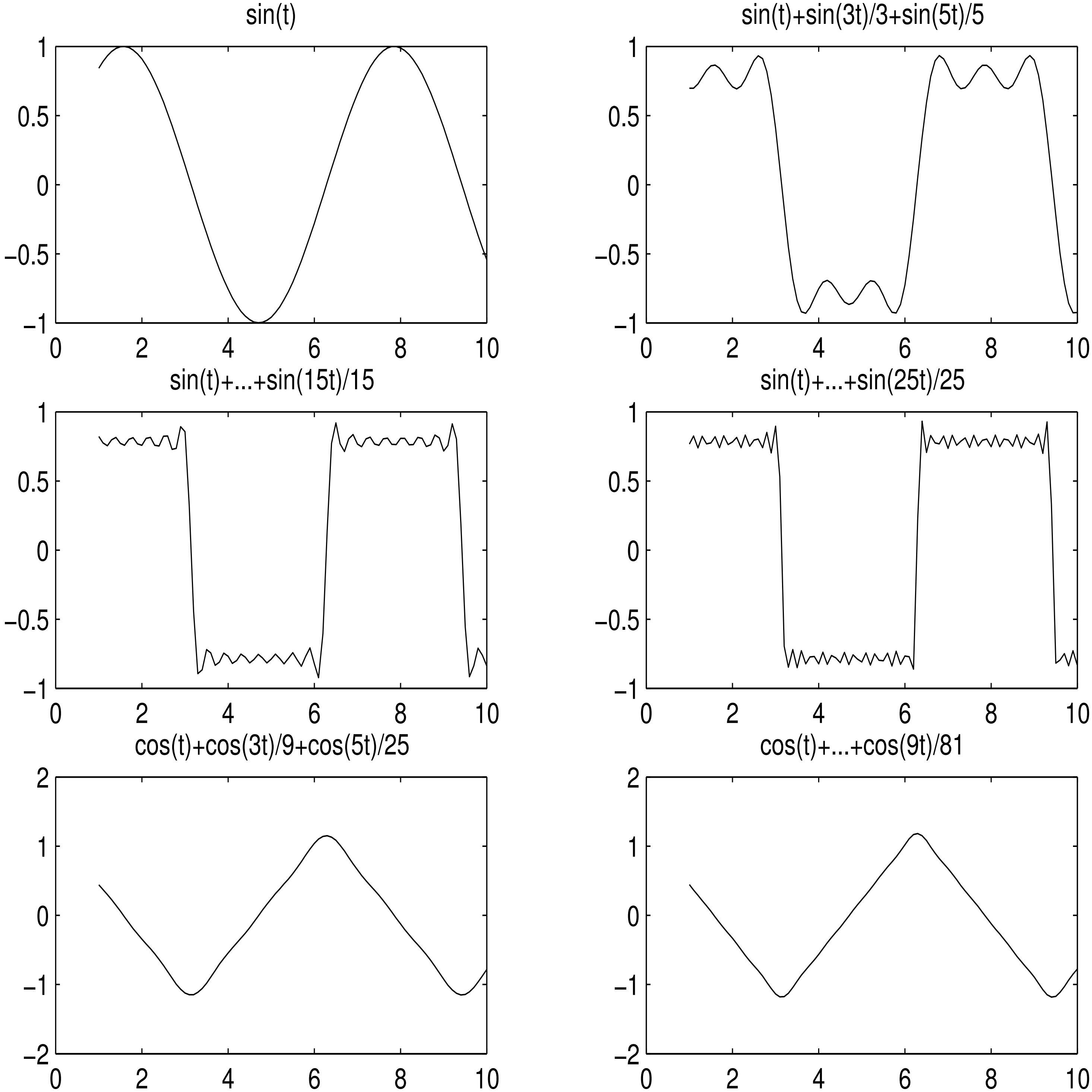
$$\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)dt = \int_{t_0}^{t_0+T} a_0\cos(n\omega_0 t)dt$$
$$+\int_{t_0}^{t_0+T} a_1\cos(n\omega_0 t)\cos(\omega_0 t)dt + \int_{t_0}^{t_0+T} a_2\cos(n\omega_0 t)\cos(2\omega_0 t)dt+\ldots$$
$$+\int_{t_0}^{t_0+T} b_1\cos(n\omega_0 t)\sin(\omega_0 t)dt + \int_{t_0}^{t_0+T} b_2\cos(n\omega_0 t)\sin(2\omega_0 t)dt+\ldots$$

$=0+0+\ldots+0+a_n\frac{T}{2}+0+\ldots$ from which the a_n formula follows.

The formulae for a_0 and b_n follow similarly. QED.

COMMENTS:

1. t_0 is arbitrary; all integrals are over one period.
2. $\omega_0=\frac{2\pi}{T}$ is angular frequency in $\frac{\text{RADIAN}}{\text{SECOND}}$; this is $\frac{1}{T}$ Hertz.
3. The sinusoid at frequency ω_0 is called the *fundamental*; the sinusoid at frequency $n\omega_0$ is the n^{th} *harmonic*. Harmonics are also called *overtones* (not used much anymore). Some say the n^{th} harmonic is at frequency $(n+1)\omega_o$.
4. The more terms (i.e., harmonics) we keep in the Fourier series, the better the approximation the truncated series is to $x(t)$.
5. If $x(t)$ has a discontinuity at $t=t_1$, the Fourier series converges to $\frac{1}{2}(x(t_1^-)+x(t_1^+))$, where $x(t^-)$ and $x(t^+)$ are the values of $x(t)$ on either side of the discontinuity.


sin(t)
sin(t)+sin(3t)/3+sin(5t)/5
sin(t)+...+sin(15t)/15
sin(t)+...+sin(25t)/25
cos(t)+cos(3t)/9+cos(5t)/25
cos(t)+...+cos(9t)/81
1
0.5
0
−0.5
−1
2
−2
0
2
4
6
8
10