EXAMPLES OF FIR DIGITAL FILTERS

FIR: FIR=Finite Impulse Response: $h[n]$ has finite *duration* and *support*.
Form: $y[n] = b_o x[n] + b_1 x[n-1] + b_2 x[n-2] + \ldots + b_M x[n-M]$. Order=$M$.
h[n]: $h[n] = b_n$ for $0 \le n \le M$; $h[n] = 0$ otherwise. Assume $h[n]$ is real here.

Frequency response: $H(e^{j\omega}) = b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega} + \ldots + b_M e^{-jM\omega}$.
$e^{j\omega_o n} \to \boxed{FIR} \to H(e^{j\omega_o}) e^{j\omega_o n}$, implying
$\cos(\omega_o n) \to \boxed{FIR} \to |H(e^{j\omega_o})| \cos(\omega_o n + arg[H(e^{j\omega_o})])$

Properties of $H(e^{j\omega})$: $H(e^{j\omega})$ **periodic** in ω with period 2π.
Conjugate symmetry: $H(e^{-j\omega}) = H^*(e^{j\omega})$.
Implies: **Gain** is **even** function: $|H(e^{j\omega})| = |H(e^{-j\omega})|$.
Phase is **odd** function: $arg[H(e^{-j\omega})] = -arg[H(e^{j\omega})]$.

Notch filter: $h[n] = \{\underline{1}, -2\cos(\omega_o), 1\} \to H(e^{j\omega}) = [2\cos(\omega) - 2\cos(\omega_o)] e^{-j\omega}$.
Implement using $y[n] = x[n] - 2\cos(\omega_o) x[n-1] + x[n-2]$.
Does: Eliminates component at a single frequency ω_o from input.
Why? Eliminate interference at a single frequency, e.g., 60 Hz "hum."

Comb filter: $h[n] = \{\underline{1}, 1, 1 \ldots 1\} \to H(e^{j\omega}) = \frac{e^{-j\omega(M+1)} - 1}{e^{-j\omega} - 1} = \frac{\sin(\frac{M+1}{2}\omega)}{\sin(\frac{\omega}{2})} e^{-j\omega M/2}$.
Implement using $y[n] = x[n] + x[n-1] + \ldots + x[n-M]$.
Does: Eliminates components at frequencies $\pm\frac{2\pi}{M+1}, \pm\frac{4\pi}{M+1} \ldots \pm \frac{M\pi}{M+1}$.
Why? Eliminate interference at **harmonics** of a single frequency $2\pi/(M+1)$.
e.g., Sawtooth wave interference with period $(M+1)$ has these harmonics.

Ideal LPF: $h[n] = \frac{\sin(Bn)}{\pi n} \to H(e^{j\omega}) = 1$ for $0 < |\omega| < B; = 0$ for $B < |\omega| < \pi$.
Implement using $y[n] = \sum_{i=-\infty}^{\infty} \frac{\sin(Bi)}{\pi i} x[n-i]$; truncate the sum.
Does: Low-pass filter; eliminates frequencies above B (periodic; period 2π).
Why? Filter out high-frequency noise from a low-frequency signal (smooth).

APPLICATION TO ELECTROCARDIOGRAM (HEART) SIGNALS

Given: $x(t)$=EKG signal; periodic (you hope!) at 60 $\frac{\text{BEATS}}{\text{MINUTE}}$=1 $\frac{\text{BEAT}}{\text{SECOND}}$.
Goal: Filter out 60 Hz interference from electrical outlet wires in lab.

Signal $x(t)$ period=1 second→fundamental=1 Hz→harmonics=2,3… Hz.
Huh? $x(t) = \underbrace{c_0}_{\text{DC}} + \underbrace{c_1\cos(2\pi t + \theta_1)}_{\text{FUNDAMENTAL}} + \underbrace{c_2\cos(4\pi t + \theta_2)}_{\text{1ST HARMONIC}} + \underbrace{c_3\cos(6\pi t + \theta_3)}_{\text{2ND HARMONIC}} \ldots$

Sample at 1 kHz $\to \Delta = 0.001$ second $\to t = 0.001n \to x[n] = x(t = 0.001n)$.
Spectrum of sampled signal $x[n] = x(0.001n)$ is periodic in f with period 1 kHz.

Notch $y[n] = x[n] - 2\cos(2\pi\frac{60}{1000}) x[n-1] + x[n-2]$ eliminates 60 Hz "hum."

ARMA DIFFERENCE EQUATIONS

MA: $y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_q x[n-q]$ (Moving Average)

Huh? Present output=weighted average of $q+1$ *most recent* inputs.

Note: Equivalent to $y[n] = b[n] * x[n]$ where $b[k] = b_k, 0 \leq k \leq q$.

AR: $y[n] + a_1 y[n-1] + \ldots + a_p y[n-p] = x[n]$ (AutoRegression)

Huh? Present output=weighted sum of p *most recent* **outputs**.

Note: Compute $y[n]$ *recursively* from its p most recent values.

ARMA:

$$\underbrace{\sum_{i=0}^{p} a_i y[n-i]}_{\text{AUTOREGRESSIVE}} = \underbrace{\sum_{i=0}^{q} b_i x[n-i]}_{\text{MOVING AVERAGE}} \quad \begin{cases} \text{MA} \Leftrightarrow \text{FIR (Finite Impulse)} \\ \text{AR} \rightarrow \text{IIR (Infinite Impulse)} \\ \text{ARMA} \rightarrow \text{IIR (Response)} \end{cases}$$

Note: Difference equation$\sim$*differential equation* in continuous time:

$$\left(\frac{d^p}{dt^p} + a_1 \frac{d^{p-1}}{dt^{p-1}} + \ldots + a_p\right) y(t) = \left(\frac{d^q}{dt^q} + b_1 \frac{d^{q-1}}{dt^{q-1}} + \ldots + b_q\right) x(t).$$

Note: Coefficients a_i and b_i are *not directly* analogous here to a_i and b_i above.

Note: Linear time-invariant in both cases since all coefficients indpt. of time.

EXAMPLES OF IIR FILTERS

1. $h[n] = ba^n u[n] \Leftrightarrow H(z) = b\frac{z}{z-a}\frac{z^{-1}}{z^{-1}} \Leftrightarrow y[n] - ay[n-1] = bx[n]$ [1^{st}-order].

2. $h[n] = (c_1 p_1^n + c_2 p_2^n)u[n] \Leftrightarrow H(z) = \frac{c_1 z}{z-p_1} + \frac{c_2 z}{z-p_2} = \frac{(c_1+c_2)z^2 - (c_1 p_2 + c_2 p_1)z}{z^2 - (p_1+p_2)z + (p_1 p_2)}$

$\Leftrightarrow y[n] - (p_1+p_2)y[n-1] + (p_1 p_2)y[n-2] = (c_1+c_2)x[n] - (c_1 p_2 + c_2 p_1)x[n-1]$

2a. $h[n] = 2\cos(\omega_o n)u[n] = (e^{j\omega_o n} + e^{-j\omega_o n})u[n]$ [$(2,1)^{nd}$-order ARMA]

$\Leftrightarrow y[n] - 2\cos(\omega_o)y[n-1] + y[n-2] = 2x[n] - 2\cos(\omega_o)x[n-1]$.

2b. $h[n] = 6(\frac{1}{2})^n u[n] + 8(\frac{1}{3})^n u[n]$ can be implemented by:[$8(\frac{1}{2}) + 6(\frac{1}{3}) = 6$]

$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 14x[n] - 6x[n-1]$ [$(2,1)^{nd}$-order ARMA].

SO WHY BOTHER TO USE IIR FILTERS?

Goal: Eliminate 60 Hz "hum" in a continuous-time signal sampled at 250 Hz.

FIR: Notch: $y[n] = x[n] - 2\cos(2\pi\frac{60}{250})x[n-1] + x[n-2]$. See below left.

IIR: ARMA: $y[n] - 1.98\cos(2\pi\frac{60}{250})y[n-1] + 0.98y[n-2]$ [$(2,2)^{nd}$-order]

$= x[n] - 2\cos(2\pi\frac{60}{250})x[n-1] + x[n-2]$. See below right.

Point: The IIR filter has a *much* sharper frequency response than the FIR filter.