CONCEPTS BEHIND THE DISCRETE FOURIER TRANSFORM (DFT)

NOTE: See *DFT: Discrete Fourier Transform* for more details.

Given: $x[n]$ is a discrete-time signal with period N: $x[n] = x[n+N]$ for all n.

DFT: $x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$ where $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$.

- Fastest-oscillating discrete-time sinusoid: $\omega = \pi \rightarrow \cos(\pi n) = (-1)^n$.

$\rightarrow$ Fourier series of discrete-time periodic signal has **finite** number of terms, with frequencies

$\{0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N} \dots (N-1)\frac{2\pi}{N}\} \Leftrightarrow \{0, \pm\frac{2\pi}{N}, \pm 2\frac{2\pi}{N} \dots \pm \frac{N-1}{2}\frac{2\pi}{N}, [\pi?]\}$.

Huh? If N even, the component with the highest frequency is $\omega = \pi$.
If N odd, the component with the highest frequency is $\omega = \frac{N-1}{N}\pi$.

- If $x[n]$ is real, then $X_{N-k} = X_k^*$ (conjugate symmetry).
- $X_0 = \frac{1}{N}(x[0] + x[1] + \dots + x[N-1])$=mean value of $x[n]$.
- If N is even, $X_{N/2} = \frac{1}{N}(x[0] - x[1] + x[2] - x[3] + \dots - x[N-1])$.

SIMPLE EXAMPLE WITH N=4:

Given: $x[n] = \{\dots 24, 8, 12, 16, \underline{24}, 8, 12, 16, 24, 8, 12, 16 \dots\}$. Period=N=4.
Goal: Compute DFT=Fourier series expansion of discrete-time periodic $x[n]$.

- NOTE: $e^{-j\frac{2\pi}{4}1} = -j; e^{-j\frac{2\pi}{4}2} = -1; e^{-j\frac{2\pi}{4}3} = +j$.

1. $X_0 = \frac{1}{4}(24 + 8 + 12 + 16) = 15$. Note this is real.
2. $X_2 = \frac{1}{4}(24 - 8 + 12 - 16) = 03$. Note this is real.
3. $X_1 = \frac{1}{4}(24 + 8(-j) + 12(-1) + 16(+j)) = 3 + 2j$.
4. $X_3 = \frac{1}{4}(24 + 8(+j) + 12(-1) + 16(-j)) = 3 - 2j = X_1^*$.

Then: $x[n] = (15)e^{j0n} + (3+2j)e^{j\frac{2\pi}{4}n} + (03)e^{j\frac{2\pi}{4}2n} + (3-2j)e^{j\frac{2\pi}{4}3n}$.

Line spectrum is **periodic** with components at: $\{0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi \dots\}$.

Using: $3 + 2j = 3.6e^{j33.7^o}; e^{j\pi n} = \cos(\pi n); e^{j\frac{2\pi}{4}3n} = e^{-j\frac{2\pi}{4}n}$, simplifies to:

$x[n] = 15 + 7.2\cos(\frac{\pi}{2}n + 33.7^o) + 3\cos(\pi n)$. Don't double at $\omega = 0, \pi$.

PARSEVAL'S THEOREM: POWER IS CONSERVED

Power: $\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1}|X_k|^2$=average power of periodic $x[n]$.

Time: $15^2 + |3+2j|^2 + 3^2 + |3-2j|^2 = 260$ since $|3+2j|^2 = 13$.
Freq: $\frac{1}{4}(24^2 + 8^2 + 12^2 + 16^2) = 260$. They are equal!

EXAMPLE OF DISCRETE-TIME FOURIER SERIES (DFT):

What? Like continuous time, except *finite* #terms→*exact* representation.
Below: $x[n] = c_1 \cos(\omega_o n) + c_2 \cos(2\omega_o n) + \ldots + c_8 \cos(8\omega_o n)$=even function
where: $\omega_o = \frac{2\pi}{N} = \frac{2\pi}{17}$ and $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} = \frac{1}{17}\frac{\sin(9\pi k/17)}{\sin(\pi k/17)}$.