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**THE DISCRETE FOURIER TRANSFORM (DFT)**


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**NOTE:** See *DFT: Discrete Fourier Transform* for more details.

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**Cont.** Let  $x(t) = x(t + T)$  be periodic with period= $T$  in **continuous** time.

**Time** Then  $x(t)$  can be expanded in the *continuous-time* Fourier series

**Fourier**  $x(t) = X_0 + X_1 e^{j\frac{2\pi}{T}t} + X_2 e^{j\frac{4\pi}{T}t} + \dots + X_{-1} e^{-j\frac{2\pi}{T}t} + X_{-2} e^{-j\frac{4\pi}{T}t} + \dots$

**Series** where  $X_k = \frac{1}{T} \int_{t_o}^{t_o+T} x(t) e^{-j\frac{2\pi}{T}kt} dt$  for integers  $k$  and any time  $t_o$ .

**Note: Conjugate symmetry:**  $x(t)$  real  $\Leftrightarrow X_{-k} = X_k^*$  for integers  $k$ .

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**Discrete** Let  $x[n] = x[n + N]$  be periodic with period= $N$  in **discrete** time.

**Time** Then  $x[n]$  can be expanded in the *discrete-time* Fourier series

**Fourier**  $x[n] = X_0 + X_1 e^{j\frac{2\pi}{N}n} + X_2 e^{j\frac{4\pi}{N}n} + \dots + X_{N-1} e^{j\frac{(N-1)2\pi}{N}n}$

**Series** where  $X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$  =DFT for  $k = 0 \dots N - 1$ .

**Note: Conjugate symmetry:**  $x[n]$  real  $\Leftrightarrow X_{N-k} = X_k^*$  for integers  $k$ .

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**EX:**  $x[n] = \{\dots 12, 6, 4, 6, \underline{12}, 6, 4, 6, 12, 6, 4, 6 \dots\}$ . Periodic; period  $N = 4$ .

**DFT:**  $X_0 = \frac{1}{4}(x[0] + (+1)x[1] + (+1)x[2] + (+1)x[3]) = \frac{1}{4}(12 + 6 + 4 + 6) = 7$ .

**DFT:**  $X_1 = \frac{1}{4}(x[0] + (-j)x[1] + (-1)x[2] + (+j)x[3]) = \frac{1}{4}(12 - 6j - 4 + 6j) = 2$ .

**DFT:**  $X_2 = \frac{1}{4}(x[0] + (-1)x[1] + (+1)x[2] + (-1)x[3]) = \frac{1}{4}(12 - 6 + 4 - 6) = 1$ .

**DFT:**  $X_3 = \frac{1}{4}(x[0] + (+j)x[1] + (-1)x[2] + (-j)x[3]) = \frac{1}{4}(12 + 6j - 4 - 6j) = 2$ .

**Note:**  $X_3 = X_{4-1} = X_1^* = 2^* = 2$  (although both  $X_3$  and  $X_1$  are real here).

**Note:**  $x[n]$  is a real and even function  $\Leftrightarrow X_k$  is a real and even function.

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**Then:**  $x[n] = 7 + 2e^{j\frac{\pi}{2}n} + 1e^{j\pi n} + 2e^{j\frac{3\pi}{2}n}$  (**complex exponential form**)

**Or:**  $x[n] = 7 + 4 \cos(\frac{\pi}{2}n) + 1 \cos(\pi n)$  (**trigonometric form**)

**since:**  $e^{j\frac{3\pi}{2}n} = e^{-j\frac{\pi}{2}n}$  (try it) and  $e^{j\pi n} = \cos(\pi n) = (-1)^n$ .

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**Power:** Time domain: Average power= $\frac{1}{4}(12^2 + 6^2 + 4^2 + 6^2) = 58$ .

**Parseval:** Freq. domain: Average power= $(|7|^2 + |2|^2 + |1|^2 + |2|^2) = 58$ .

**So?** Compute average power in either time domain or frequency domain.

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**So?** Consider  $x[n] \rightarrow \overline{LTI} \rightarrow y[n]$  where input  $x[n] = \{\dots 12, 6, 4, 6 \dots\}$   
and Linear Time-Invariant (LTI) system is  $y[n] - 3y[n-1] = 3x[n] + 3x[n-1]$ .

**Then:** Frequency response function= $H(e^{j\omega}) = 3 \frac{e^{j\omega} - 1}{e^{j\omega} - 3}$  (Huh? stay tuned)

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**Then:**  $H(e^{j0}) = 3 \frac{+1-1}{+1-3} = 0$ ;  $H(e^{j\pi/2}) = 3 \frac{+j-1}{+j-3} = 1.341 e^{-j0.46}$

**and:**  $H(e^{j\pi}) = 3 \frac{-1-1}{-1-3} = \frac{3}{2}$ ;  $H(e^{j3\pi/2}) = 3 \frac{-j-1}{-j-3} = 1.341 e^{j0.46}$

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**Then:**  $y[n] = (0)7 + 1.341 e^{-j0.46} 2 e^{jn\pi/2} + \frac{3}{2} 1 e^{jn\pi} + 1.341 e^{j0.46} 2 e^{jn3\pi/2}$

**and:**  $y[n] = 7(0) + 4(1.341) \cos(\frac{\pi}{2}n - 0.46) + 1(\frac{3}{2}) \cos(\pi n)$  which becomes

$y[n] = 5.366 \cos(\frac{\pi}{2}n - 0.46) + 1.5 \cos(\pi n)$ . Note DC term filtered out.

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**EXAMPLES OF DFT PROPERTIES**


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**Given:**  $x[n]$  is a discrete-time signal with period  $N$ :  $x[n] = x[n + N]$  for all  $n$ .

**DFT:**  $x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$  where  $X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$

1.  $X_0 = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$  = DC value = mean value of periodic signal  $x[n]$ .
2. Negative frequencies are second half of  $\{X_k\}$ : Use  $X_{-k} = X_{N-k}$ .
3. Matlab's **fftshift** shifts DC to the center, from the left end of plot. This makes conjugate symmetry  $X_{-k} = X_{N-k} = X_k^*$  easier to see.
4. This is EECS 206 definition;  $\frac{1}{N}$  moved everywhere else. Matlab: **fft**.

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**EX #1:** DFT $\{\underline{1}, 0, 0, 0, 0, 0, 0, 0\} = \frac{1}{8}\{1, 1, 1, 1, 1, 1, 1, 1\}$ . **Impulse** in time.

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**EX #2:** DFT $\{\underline{0}, 0, 1, 0, 0, 0, 0, 0\} = \frac{1}{8}\{1, -j, -1, j, 1, -j, -1, j\}$ . **Delayed**  $\delta[n]$ .

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**EX #3:** DFT $\{\underline{1}, 1, 1, 1, 1, 1, 1, 1\} = \{1, 0, 0, 0, 0, 0, 0, 0\}$ . **Constant** in time.

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**EX #4:** DFT $\{\underline{1}, 1, 2, 1, 1, 1, 1, 1\} = \frac{1}{8}\{9, -j, -1, j, 1, -j, -1, j\}$ . DFT is **linear**.

**Parseval:** Average power =  $\frac{11}{8} = \frac{1}{8}(1^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2)$   
 $= (\frac{1}{8})^2(9^2 + |-j|^2 + |-1|^2 + |j|^2 + |1|^2 + |-j|^2 + |-1|^2 + |j|^2)$ .

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**EX #5:** DFT $\{\cos(2\pi \frac{M}{N}n + \theta)\} = \frac{1}{2}e^{j\theta}\delta[k - M] + \frac{1}{2}e^{-j\theta}\delta[k - (N - M)]$ .

**Note:** This only works for *periodic* discrete-time sinusoids:  $\omega_o = 2\pi \frac{M}{N}$ .

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**EX:** DFT $\{\underline{24}, 8, 12, 16\} = \{15, 3 + 2j, 3, 3 - 2j\}$  (1 period of  $x[n]$  and  $X_k$ ).

$$\rightarrow x[n] = (15)e^{j0n} + (3 + 2j)e^{j(\pi/2)n} + (03)e^{j\pi n} + (3 - 2j)e^{j(3\pi/2)n}$$

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1. DFT $\{\underline{24}, 16, 12, 8\} = \{15, 3 - 2j, 3, 3 + 2j\}$ . **Reversal:**  $x[-n] \rightarrow X_k^*$ .

**Huh?**  $x[+n] = \{\dots 24, 8, 12, 16, \underline{24}, 8, 12, 16, 24, 8, 12, 16 \dots\} \rightarrow$   
 $x[-n] = \{\dots 24, 16, 12, 8, \underline{24}, 16, 12, 8, 24, 16, 12, 8 \dots\}$ .

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2. DFT $\{\underline{12}, 16, 24, 8\} = \{15, -3 - 2j, 3, 2j - 3\}$ . **Delay:**  $x[n-D] \rightarrow X_k e^{-\frac{j2\pi kD}{N}}$ .

**Huh?**  $x[n - 2] = \{\dots 12, 16, 24, 8, \underline{12}, 16, 24, 8, 12, 16, 24, 8 \dots\}$ .

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3. DFT $\{\underline{24}, -8, 12, -16\} = \{3, 3 - 2j, 15, 3 + 2j\}$ .  $x[n]e^{\frac{j2\pi nF}{N}} \rightarrow X_{k-F}$

**Huh?** "Modulate" signal means *shift* its spectrum by some frequency  $F$ .

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4. DFT $\{\underline{24}, 0, 8, 0, 12, 0, 16, 0\} = \frac{1}{2}\{15, 3 + 2j, 3, 3 - 2j, 15, 3 + 2j, 3, 3 - 2j\}$ .

**Huh?** Interpolate with zeros  $\rightarrow$  repeat and halve DFT of lower order.

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5. DFT $\{\underline{24}, 8, 12, 16, 24, 8, 12, 16\} = \{15, 0, 3 + 2j, 0, 3, 0, 3 - 2j, 0\}$ .

**Huh?** Repeat in time  $\rightarrow$  interpolate with zeros in frequency domain.