THE DISCRETE FOURIER TRANSFORM (DFT)

**NOTE**: See DFT: Discrete Fourier Transform for more details.

**Cont.** Let \( x(t) = x(t + T) \) be periodic with period \( T \) in **continuous** time.

**Time** Then \( x(t) \) can be expanded in the **continuous-time** Fourier series

**Fourier** \( x(t) = X_0 + X_1e^{j\frac{2\pi}{T}t} + X_2e^{j\frac{4\pi}{T}t} + \ldots + X_{-1}e^{-j\frac{2\pi}{T}t} + X_{-2}e^{-j\frac{4\pi}{T}t} + \ldots \)

**Series** where \( X_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)e^{-j\frac{2\pi}{T}kt}dt \) for integers \( k \) and any time \( t_0 \).

**Note**: Conjugate symmetry: \( x(t) \) real \( \iff X_{-k} = X_k^* \) for integers \( k \).

**Discrete** Let \( x[n] = x[n + N] \) be periodic with period \( N \) in **discrete** time.

**Time** Then \( x[n] \) can be expanded in the **discrete-time** Fourier series

**Fourier** \( x[n] = X_0 + X_1e^{j\frac{2\pi}{N}n} + X_2e^{j\frac{4\pi}{N}n} + \ldots + X_{N-1}e^{j\frac{(N-1)2\pi}{N}n} \)

**Series** where \( X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \text{DFT} \) for \( k = 0 \ldots N - 1 \).

**Note**: Conjugate symmetry: \( x[n] \) real \( \iff X_{N-k} = X_k^* \) for integers \( k \).

**EX**: \( x[n] = \{ \ldots 12, 6, 4, 6, 12, 6, 4, 6, 12, 6, 4, 6 \ldots \} \). Periodic; period \( N = 4 \).

**DFT**: \( X_0 = \frac{1}{4}[(x[0] + (+1)x[1] + (+1)x[2] + (+1)x[3])] = \frac{1}{4}(12 + 6 + 4 + 6) = 7 \).

**DFT**: \( X_1 = \frac{1}{4}[(x[0] + (-j)x[1] + (-1)x[2] + (+j)x[3])] = \frac{1}{4}(12 - 6j - 4 + 6j) = 2 \).

**DFT**: \( X_2 = \frac{1}{4}[(x[0] + (-1)x[1] + (+1)x[2] + (-1)x[3])] = \frac{1}{4}(12 - 6 + 4 - 6) = 1 \).

**DFT**: \( X_3 = \frac{1}{4}[(x[0] + (+j)x[1] + (-1)x[2] + (-j)x[3])] = \frac{1}{4}(12+6j-4-6j) = 2 \).

**Note**: \( X_3 = X_{4-1} = X_1^* = 2^* = 2 \) (although both \( X_3 \) and \( X_1 \) are real here).

**Note**: \( x[n] \) is a real and even function \( \iff X_k \) is a real and even function.

\( \text{Then: } x[n] = 7 + 2e^{j\frac{\pi}{2}} + 1e^{j\pi n} + 2e^{j\frac{3\pi}{2}} \) (complex exponential form)

Or: \( x[n] = 7 + 4 \cos(\frac{\pi}{2} n) + 1 \cos(\pi n) \) (trigonometric form)

since: \( e^{j\frac{2\pi}{N}n} = e^{-j\frac{\pi}{2} n} \) (try it) and \( e^{j\pi n} = \cos(\pi n) = (-1)^n \).

**Power**: Time domain: Average power = \( \frac{1}{4}(12^2 + 6^2 + 4^2 + 6^2) = 58 \).

**Parseval**: Freq. domain: Average power = \( (|7|^2 + |2|^2 + |1|^2 + |2|^2) = 58 \).

**So?** Compute average power in either time domain or frequency domain.

**So?** Consider \( x[n] \to [\text{LTI}] \to y[n] \) where input \( x[n] = \{ \ldots 12, 6, 4, 6 \ldots \} \)

and Linear Time-Invariant (LTI) system is \( y[n] = 3y[n-1] = 3x[n] + 3x[n-1] \).

**Then**: Frequency response function = \( H(e^{j\omega}) = 3e^{j\omega - \frac{1}{3}} \) (Huh? stay tuned)

**Then**: \( H(e^{j0}) = 3 + \frac{1}{3} - \frac{1}{3} = 0 \); \( H(e^{j\pi/2}) = 3 + \frac{j}{3} - \frac{j}{3} = 1.341e^{-j0.46j} \)

and: \( H(e^{j\pi}) = 3 - \frac{1}{3} - \frac{1}{3} = \frac{2}{3} \); \( H(e^{j3\pi/2}) = 3 - \frac{j}{3} - \frac{j}{3} = 1.341e^{j0.46j} \)

**Then**: \( y[n] = (0)7 + 1.341e^{-j0.46j}2e^{jn\pi/2} + \frac{3}{2}1e^{jn\pi} + 1.341e^{j0.46j}2e^{jn3\pi/2} \)

**And**: \( y[n] = 7(0) + 4(1.341) \cos(\frac{\pi}{2} n - 0.46) + 1(\frac{3}{2}) \cos(\pi n) \) which becomes \( y[n] = 5.366 \cos(\frac{\pi}{2} n - 0.46) + 1.5 \cos(\pi n) \). Note DC term filtered out.
**EXAMPLES OF DFT PROPERTIES**

**Given:** \( x[n] \) is a discrete-time signal with period \( N \): \( x[n] = x[n + N] \) for all \( n \).

**DFT:** \( x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N} \) where \( X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \)

1. \( X_0 = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \) is the DC value, the mean value of the periodic signal \( x[n] \).
2. Negative frequencies are second half of \( \{X_k\} \): Use \( X_{-k} = X_{N-k} \).
3. Matlab’s `fftshift` shifts DC to the center, from the left end of plot. This makes conjugate symmetry \( X_{-k} = X_{N-k} = X_k^* \) easier to see.
4. This is EECS 206 definition; \( \frac{1}{N} \) moved everywhere else. Matlab: `fft`.

| EX #1 | DFT\{1, 0, 0, 0, 0, 0, 0, 0\} = \( \frac{1}{8} \{1, 1, 1, 1, 1, 1, 1, 1\} \). **Impulse** in time. |
| EX #2 | DFT\{0, 0, 1, 0, 0, 0, 0, 0\} = \( \frac{1}{8} \{1, -1, -1, 1, 1, -1, -1, j\} \). **Delayed** \( \delta[n] \). |
| EX #3 | DFT\{1, 1, 1, 1, 1, 1, 1, 1\} = \( \frac{1}{8} \{1, 0, 0, 0, 0, 0, 0, 0\} \). **Constant** in time. |
| EX #4 | DFT\{1, 1, 1, 1, 1, 1, 1, 1\} = \( \frac{1}{8} \{9, -j, 1, -j, 1, -j, -1, j\} \). DFT is **linear**. |

**Parseval:** Average power = \( \frac{11}{8} = \frac{1}{8}(1^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 2^2) \)

= \( (\frac{1}{2})^2(9^2 + |j|^2 + |1|^2 + |j|^2 + |1|^2 + |j|^2 + |1|^2 + |j|^2) \).

| EX #5 | DFT\{cos(2\pi \frac{M}{N} n + \theta)\} = \( \frac{1}{2} e^{j\theta} \delta[k - M] + \frac{1}{2} e^{-j\theta} \delta[k - (N - M)] \). |

**Note:** This only works for **periodic** discrete-time sinusoids: \( \omega_o = 2\pi \frac{M}{N} \).

| EX | DFT\{24, 8, 12, 16\} = \{15, 3 + 2j, 3, 3 - 2j\} (1 period of \( x[n] \) and \( X_k \)). |

\[ x[n] = (15)e^{j0n} + (3 + 2j)e^{j(\pi/2)n} + (03)e^{j\pi n} + (3 - 2j)e^{j(3\pi/2)n} \].

1. DFT\{24, 16, 12, 8\} = \{15, 3 - 2j, 3, 3 + 2j\}. **Reversal:** \( x[-n] \rightarrow X_k^* \).

Huh? \( x[+n] = \{\ldots 24, 8, 12, 16, 24, 8, 12, 16, 24, 8, 12, 16 \ldots \} \rightarrow \)

\[ x[-n] = \{\ldots 24, 16, 12, 8, 24, 16, 12, 8, 24, 16, 12, 8 \ldots \} \rightarrow \]

2. DFT\{12, 16, 24, 8\} = \{15, -3 - 2j, 3, 3 - 2j\}. **Delay:** \( x[n-D] \rightarrow X_k e^{-j\frac{2\pi nkD}{N}} \).

Huh? \( x[n-2] = \{\ldots 12, 16, 24, 8, 12, 16, 24, 8, 12, 16, 24, 8, 12 \ldots \} \).

3. DFT\{24, -8, 12, -16\} = \{3, 3 - 2j, 15, 3 + 2j\}. \( x[n] e^{j\frac{2\pi nP}{N}} \rightarrow X_{k-F} \).

Huh? ”Modulate” signal means **shift** its spectrum by some frequency \( F \).

4. DFT\{24, 0, 8, 0, 12, 0, 16, 0\} = \( \frac{1}{2} \{15, 3 + 2j, 3, 3 - 2j, 15, 3 + 2j, 3, 3 - 2j\} \).

Huh? Interpolate with zeros→repeat and halve DFT of lower order.

5. DFT\{24, 8, 12, 16, 24, 8, 12, 16\} = \{15, 0, 3 + 2j, 0, 3, 0, 3 - 2j, 0\}.

Huh? Repeat in time→interpolate with zeros in frequency domain.