CORRELATION BETWEEN SINUSOIDAL SIGNALS

Given: $x(t) = A\cos(\omega t + \theta_x)$ and $y(t) = B\cos(\omega t + \theta_y), -\infty < t < \infty$.
Goal: To determine the *phase difference* $|\theta_x - \theta_y|$ from data $x(t)$ and $y(t)$.

Soln: Compute the $\begin{smallmatrix}\text{CORRELATION}\\ \text{COEFFICIENT}\end{smallmatrix} = \rho_{xy} = C_N(x,y) = C(x,y)/\sqrt{E(x)E(y)}$
where: $C(x,y) = \int_0^T x(t)y(t)dt$ and $E(x) = C(x,x)$ and $T = \frac{2\pi}{\omega}$=period.
Then: $\rho(x,y) = C_N(x,y) = \cos(\theta_x - \theta_y) \rightarrow |\theta_x - \theta_y|$.

Proof: $E(x) = \int_0^T A^2\cos^2(\omega t + \theta_x) = \int_0^T \frac{A^2}{2}(1 + \cos(2\omega t + 2\theta_x)) = T\frac{A^2}{2}$.
using: $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$. Compare to **rms** derivation.

$C(x,y) = \int_0^T AB\cos(\omega t + \theta_x)\cos(\omega t + \theta_y)dt = \frac{AB}{2}\int_0^T \cos(2\omega t + \theta_x + \theta_y)dt$
$+\frac{AB}{2}\int_0^T \cos(\theta_x - \theta_y)dt = T\frac{AB}{2}\cos(\theta_x - \theta_y)$
using: $\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$ and M(sinusoid)=0.

Then: $\rho_{xy} = C_N(x,y) = \frac{C(x,y)}{\sqrt{E(x)E(y)}} = \frac{(TAB/2)\cos(\theta_x - \theta_y)}{\sqrt{(TA^2/2)(TB^2/2)}} = \cos(\theta_x - \theta_y)$.
Note: (1) $\theta_x = \theta_y \Leftrightarrow \rho_{xy} = 1$; (2) $|\theta_x - \theta_y| = \frac{\pi}{2} \Leftrightarrow \rho_{xy} = 0$:
The sin and cos functions are *orthogonal* over one period.

Given: $x(t) = A\cos(2\pi f_x t)$ and $y(t) = B\cos(2\pi f_y t), -\infty < t < \infty$.
Goal: To determine the *frequency difference* $|f_x - f_y|$ from data $x(t)$ and $y(t)$.

Soln: $\begin{smallmatrix}\text{CORRELATION}\\ \text{COEFFICIENT}\end{smallmatrix} = \rho_{xy} = C_N(x,y) = \frac{C(x,y)}{\sqrt{E(x)E(y)}} = sinc((f_x - f_y)T)$
where: $sinc(x) = \frac{\sin(\pi x)}{\pi x}$ has peak at $x = 0$; smaller at half-integer x.

Proof: $E(x) = T\frac{A^2}{2}$ and $E(y) = T\frac{B^2}{2}$ as above. But now we have:
$C(x,y) = AB\int_{-T/2}^{T/2}\cos(2\pi f_x t)\cos(2\pi f_y t)dt = \frac{AB}{2}\int_{-T/2}^{T/2}\cos(2\pi(f_x + f_y)t)dt$
$+\frac{AB}{2}\int_{-T/2}^{T/2}\cos(2\pi(f_x - f_y)t)dt = T\frac{AB}{2}[sinc((f_x + f_y)T) + sinc((f_x - f_y)T)]$
using: $\frac{1}{T}\int_{-T/2}^{T/2}\cos(\omega t)dt = \frac{\sin(\omega t)}{\omega T}\Big|_{-T/2}^{T/2} = \frac{\sin(\omega T/2)}{\omega T/2} = sinc(fT) \quad (\omega = 2\pi f)$.

Now: $(f_x + f_y) >> (f_x - f_y) \rightarrow sinc((f_x + f_y)T) << sinc((f_x - f_y)T)$.
Then: $\rho_{xy} = C_N(x,y) = \frac{C(x,y)}{\sqrt{E(x)E(y)}} = \frac{(TAB/2)\mathrm{sinc}(f_x - f_y)T}{\sqrt{(TA^2/2)(TB^2/2)}} = sinc(f_x - f_y)T$.
Note: (1) $f_x = f_y \Leftrightarrow \rho_{xy} = 1$; (2) Including phase in this $\rightarrow$mess.