

Measure and cost dependent properties of information structures

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Why are information structures useful?

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- Info structures capture the design difficulties of decentralized control

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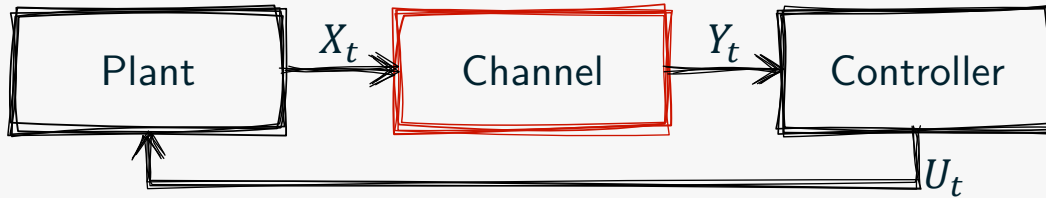
- Info structures capture the design difficulties of decentralized control
- Classical info structures are centralized systems, hence easy to design
- Non-classical info structures are decentralized systems, hence hard to design

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- Info structures capture the design difficulties of decentralized control
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Is this really true? Can we have two systems with identical information structures that behave differently?

A controller with no memory



- State Equation: $X_{t+1} = f_t(X_t, U_t, W_t)$
- Observation Equation: $Y_t = h_t(X_t, N_t)$
- **Controller with no memory:** $U_t = g_t(Y_t)$

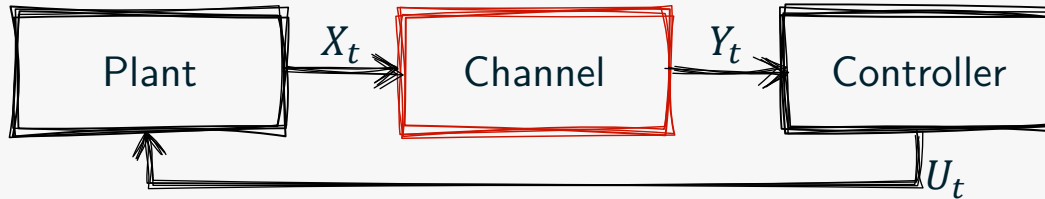
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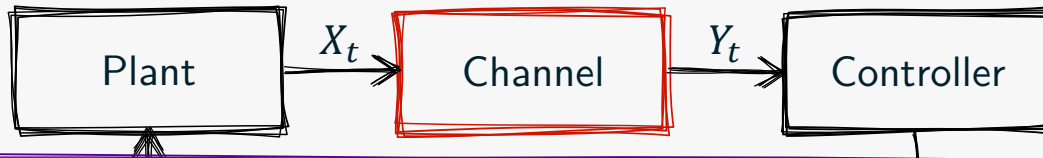


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Non-classical
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When the channel is noiseless, the system
is an MDP --- **a centralized system**

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Two systems with identical info structures

Perfect observations \Rightarrow centralized

Imperfect observations \Rightarrow decentralized

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We present a generalization of information structures, which we call ***P*-generalization**, that captures the usefulness of information.

This generalization depends on the **coupling of the cost function** and the **independence properties of the probability measure**

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- Defined a P -generalization of an info structure
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- Implications: Follow a two step approach
 - ▶ Define info structure in the usual manner (keeps analysis simple)
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We get the solution technique for P -generalized info structure for free!
- Present coupled dynamic programs to find pbpo solution of quasiclassical info structures
 - ▶ Works for non-linear systems
 - ▶ Need to only solve parametric optimization problem

Outline of the paper

- Model
- Information Structures
- P -generalization of info structures
- Coupled dynamic programs for quasiclassical info structure
- Example

The intrinsic model

Originally proposed by Witsenhausen, 1971 and 1975

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■ **Objective:** Choose (g_1, \dots, g_N) to minimize expected cost

Salient Features

Agents are coupled in two ways:

■ Coupling through dynamics

▶ D_n^* : set of agents that can influence the observations of agent n

▶ $m \in D_n^* \Rightarrow$ there exist $m = m_0, m_1, \dots, m_\ell = n$ such that

$$m_{i-1} \in D_{m_i}, \quad i = 1, \dots, \ell$$

■ Coupling through cost

C_n^* : agents coupled to agent n through cost

$$C_n^* = \bigcup_{k=1}^K C_k \mathbb{1}\{n \in C_k\}$$

Information Structures

- Information Structure

Collection of **information** known to each agent

Information Structures

■ Information Structure

Collection of **information** known to each agent

■ Classification of info structures

▶ **Classical info structure**

Each agent knows the **data** available to all agents that act before it

▶ **Quasiclassical info structure**

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▶ Strictly classical info structures

Each agent . . . **data and control actions** . . .

▶ Strictly quasiclassical info structure

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Expansion of info structures

- Classical expansion of info structure

A new system obtained by

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- Let SC be the classical expansion of C . SC is strictly classical.
- Find optimal policy g for SC (using dynamic programming)

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Question: Instead of a classical system, can we start with a more relaxed system such that this procedure still works?

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Nevertheless, we can design for classical info structure (not strict) as follows:

P -classical info structure:

$$\text{Let } Q_n := \sum_{k=1}^K \rho_k(\omega, U_{C_k}) \mathbb{1}_{\{\{n \in C_k\} \cup \{\exists m \in C_k : n \in D_m^*\}\}}.$$

Then, an info structure is P -classical if

$$\mathbb{E}\{Q_n \mid Y_n, U_n\} = \mathbb{E}\{Q_n \mid Y_{[n]}, U_{[n]}\}$$

Question: Instead of a classical system, can we start with a more relaxed system such that this procedure still works?

The main idea (2)

We ask a similar question for quasiclassical info structures.

- What is the most relaxed info structure that we can start with such that
 - ▶ if we take its quasiclassical expansion
 - ▶ find the optimal policy for the quasiclassical expansion
 - ▶ then, can find a corresponding optimal policy that is implementable in the original system

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- **Difficulty:** No appropriate solution technique for quasiclassical systems
 - ▶ Solutions for LQG quasiclassical systems rely convexity of static LQG teams. These results do not extend to non-LQG systems.
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P -quasiclassical info structure:

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Proof outline

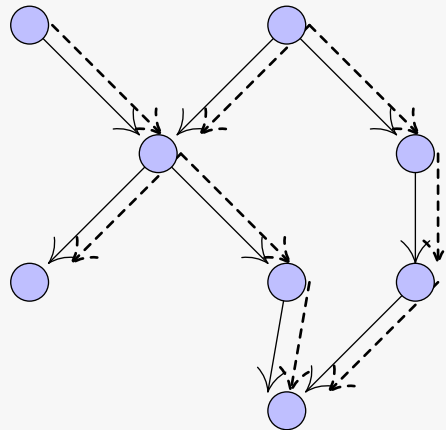
- The proof for both cases is constructive
 - ▶ Take expanded info structure
 - ▶ Find an optimal (or pbpo) policy
 - ▶ Construct a corresponding policy that is implementable in original system
- The details of each step conceptually simple, but notationally cumbersome due to generality of the model

Coupled Dynamic programs for quasiclassical info structure

Any quasiclassical system can be broken into a collection of coupled systems where each subsystem has a classical info structure

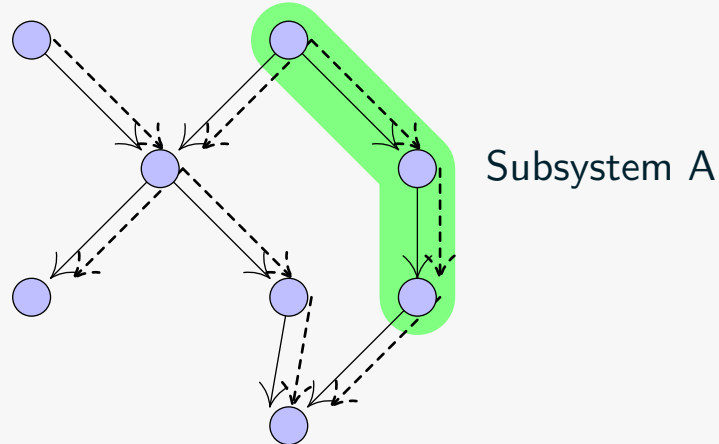
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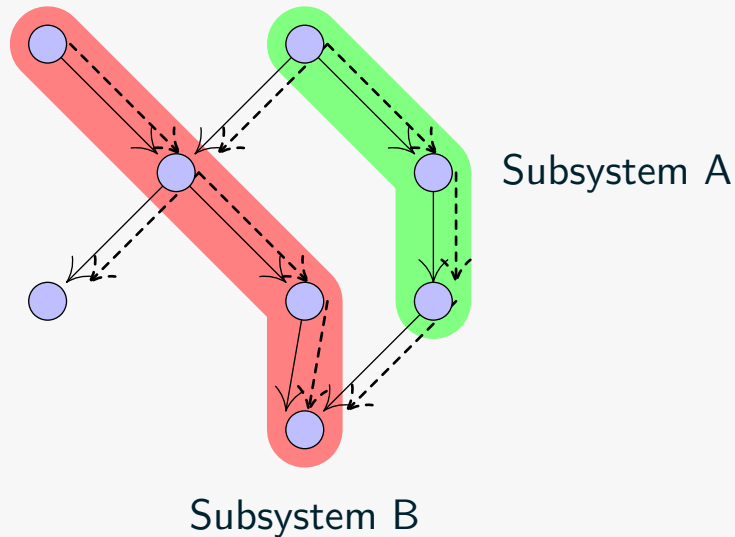
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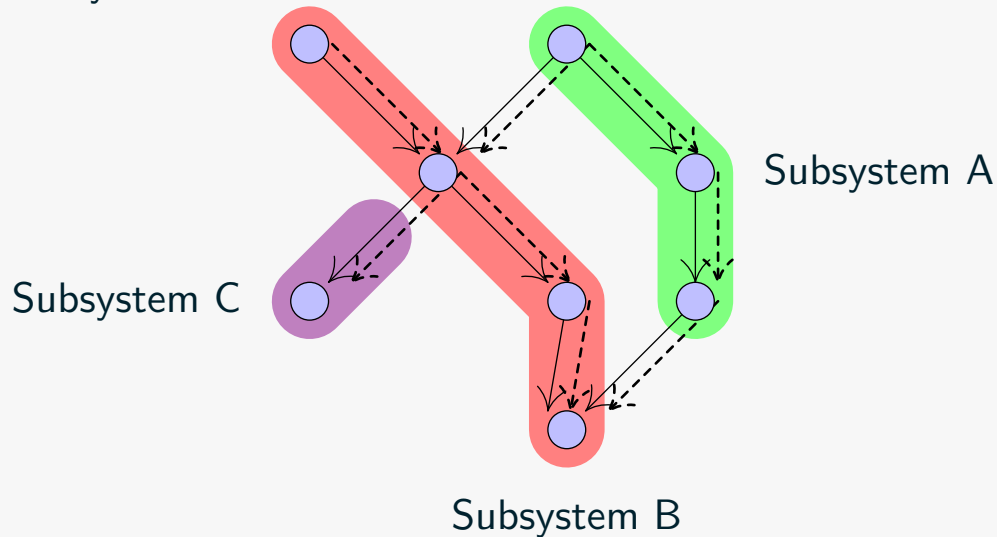
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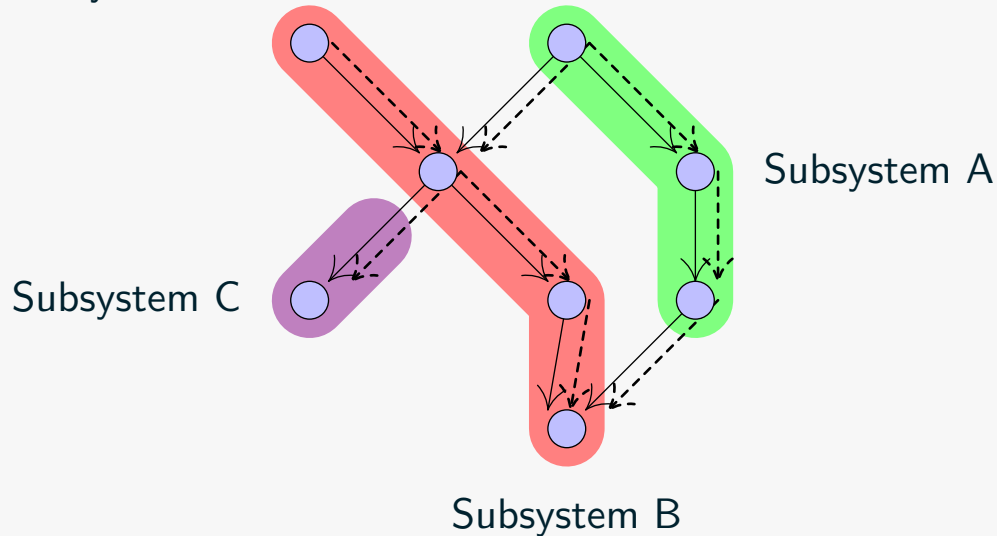
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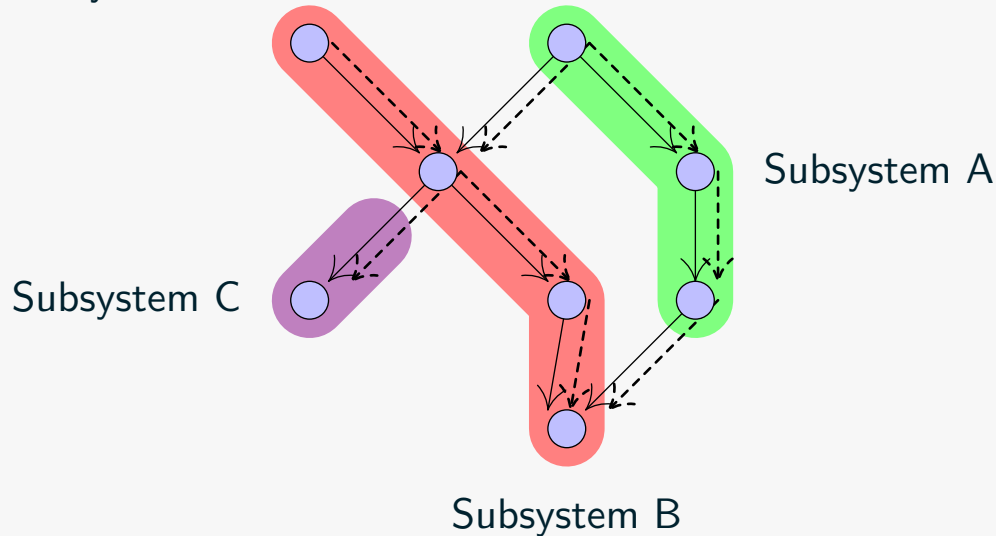
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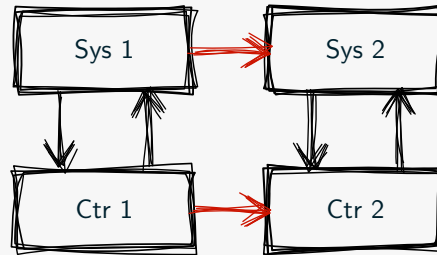
Any quasiclassical system can be broken into a collection of coupled systems where each subsystem has a classical info structure



- Subsystems A, B, and C are classical
- Write a DP for each subsystem and solve them iteratively

Idea originally proposed in Teneketzis and Ho, 1987

An Example

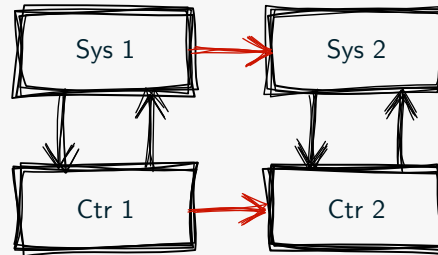


$$\begin{aligned}x_{t+1}^1 &= f^1(x_t^1, u_t^1, w_t^1) & x_{t+1}^2 &= f^2(x_t^1, x_t^2, u_t^2, w_t^2) \\y_t^1 &= h^1(x_t^1, n_t^1) & y_t^2 &= h^2(x_t^2, n_t^2) \\u_t^1 &= g_t^1(y_{[t]}^1, u_{[t-1]}^1) & u_t^2 &= g_t^2(y_{[t]}^1, y_{[t]}^2, u_{[t-1]}^1, u_{[t-1]}^2)\end{aligned}$$

Choose $G^1 := (g_1^1, \dots, g_T^1)$ and $G^2 := (g_1^2, \dots, g_T^2)$ to minimize

$$\mathbb{E} \left\{ \sum_{t=1}^T \rho(x_t^1, x_t^2, u_t^1, u_t^2) \right\}$$

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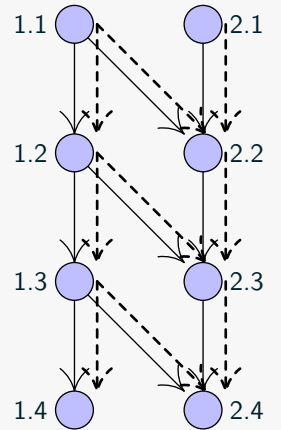
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- Quasiclassical info structure
- Non-linear dynamics
- Noisy observations

An Example



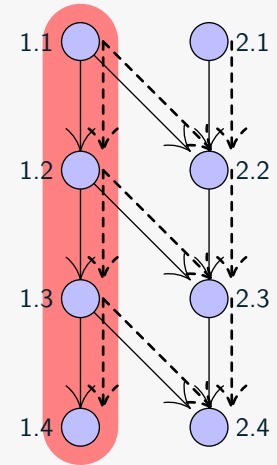
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■ Subsystem 1

Fix policy G^2 and solve for G^1

$$V_T^1(y_{[T]}^1, u_{[T-1]}^1) = \mathbb{E}\left\{\rho(x_T^1, x_T^2, u_T^1, u_T^2) \mid y_{[T]}^1, u_{[T-1]}^1\right\}$$

$$V_t^2(y_{[t]}^1, u_{[t-1]}^1) = \mathbb{E}\left\{\rho(x_t^1, x_t^2, u_t^1, u_t^2) + V_{t+1}^1(y_{[t+1]}^1, u_{[t]}^1) \mid y_{[t]}^1, u_{[t-1]}^1\right\}$$



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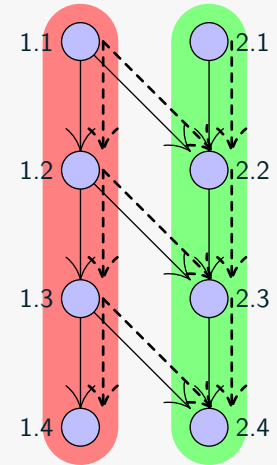
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■ Subsystem 2

Fix policy G^1 and solve for G^2

$$V_T^2(y_{[T]}^1, y_{[T]}^2, u_{[T-1]}^1, u_{[T-1]}^2) = \mathbb{E}\{\rho(x_T^2, x_T^2, u_T^2, u_T^2) | y_{[T]}^1, y_{[T]}^2, u_{[T-1]}^1, u_{[T-1]}^2\}$$

$$V_t^2(y_{[t]}^1, y_{[t]}^2, u_{[t-1]}^1, u_{[t-1]}^2) = \mathbb{E}\{\rho(x_t^2, x_t^2, u_t^2, u_t^2) + V_{t+1}^2(y_{[t+1]}^1, y_{[t+1]}^2, u_{[t]}^1, u_{[t]}^2) | y_{[t]}^1, y_{[t]}^2, u_{[t-1]}^1, u_{[t-1]}^2\}$$



Conclusion

- Defined a P -generalization of info structure

The solution technique for any info structure
is also applicable to its P -generalization

- Present coupled dynamic programs to find person by person optimal solution of quasiclassical info structures

Thank you