New (and Old) Proof Systems for Lattice Problems

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- Honest-verifier SZK $\equiv$ general SZK [GSV'98].
- SZK proofs are powerful: secure against unbounded malicious P^*, V^*.

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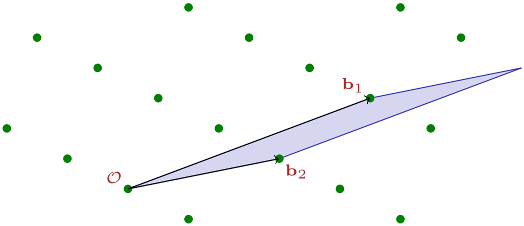
- ⋆ SZK is closed under complement [SV'97], but NISZK is not known to be.
- ⋆ NISZK is closed under complement $\iff$ NISZK $=$ SKZ [GSV'99]

Lattices

- An n-dimensional lattice $\mathcal{L} \subset \mathbb{R}^n$ is a discrete additive subgroup, generated by a (non-unique) basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$:

$$\mathcal{L} = \sum_{i=1}^{n} (\mathbb{Z} \cdot \mathbf{b}_i)$$



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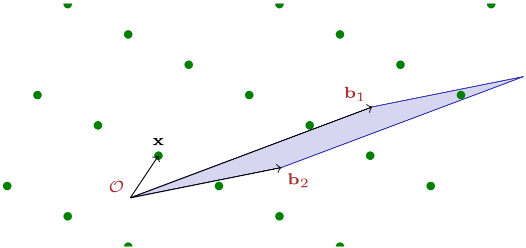

- Represent coset $\mathbf{x} + \mathcal{L} \in (\mathbb{R}^n / \mathcal{L})$ by unique $\bar{\mathbf{x}} \in (\mathbf{x} + \mathcal{L}) \cap \mathcal{P}(\mathbf{B})$.

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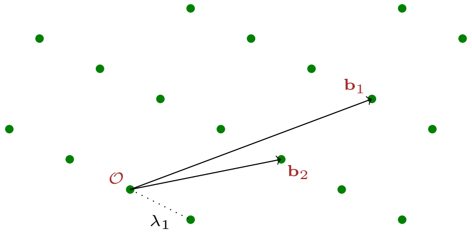

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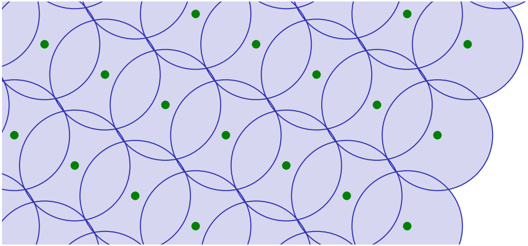

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$$\lambda_1(\mathcal{L}) = \min_{\mathbf{0} \neq \mathbf{v} \in \mathcal{L}} \|\mathbf{v}\|.$$

- Covering radius: maximum distance from the lattice

$$\mu(\mathcal{L}) = \max_{\mathbf{x} \in \mathbb{R}^n} \mathsf{dist}(\mathbf{x}, \mathcal{L}).$$

The Smoothing Parameter [MicciancioRegev'04]

- $\eta_\varepsilon(\mathcal{L})$ = minimal Gaussian 'blur' that 'smooths out' $\mathcal{L}$

  (up to error ε: think $2^{-n} \le \varepsilon \le 1/2$)

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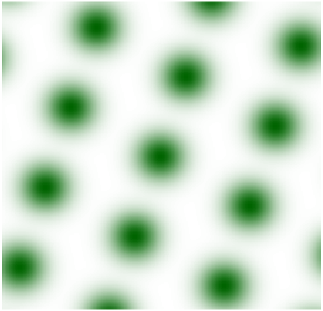

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- Algorithms for SVP and CVP [ADRS'15,ADS'15]

The Smoothing Parameter Problem [ChungDadushLiuPeikert'13]

Definition: γ-GapSPP$_\varepsilon$

- Given a lattice $\mathcal{L}$, is

$$\eta_\varepsilon(\mathcal{L}) \leq 1 \qquad \text{OR} \qquad \eta_\varepsilon(\mathcal{L}) > \gamma \quad ?$$

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Motivating Question

Are there noninteractive proof systems for GapSPP?

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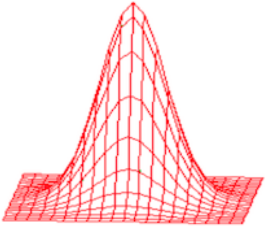
2. A reduction to EntropyApproximation $\in$ NISZK for any $\varepsilon < 1/2$.

Direct Proof of GapSPP $\in$ NISZK

Discrete Gaussians over Lattices



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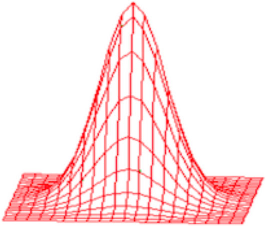

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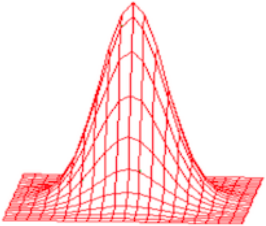

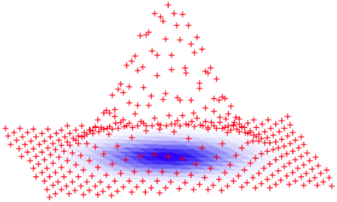

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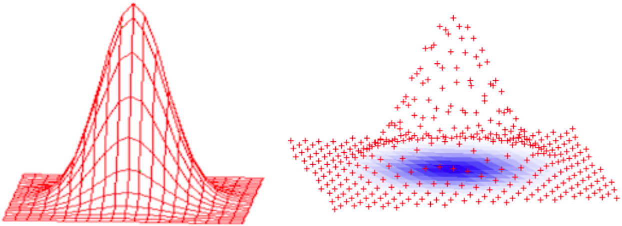

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- $D_{\mathbf{c}+\mathcal{L}}$ has Gaussian-like properties, e.g., sharp concentration bounds.

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Completeness ✓

- Suppose $\eta(\mathcal{L}) \leq 1$: implied by $\lambda_1(\mathcal{L}^*) > \sqrt{n}$.
- Then $\sigma_1(\sum \mathbf{e}_i\mathbf{e}_i^T) \leq 3m$, by matrix concentration bounds on $D_{\mathbf{c}_i+\mathcal{L}}$.

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Zero Knowledge ✓

- Suppose $\eta(\mathcal{L}) \leq 1$.
- Then cosets $\mathbf{c}_i = \mathbf{e}_i + \mathcal{L}$ are uniform* in $\mathbb{R}^n/\mathcal{L}$, and $\mathbf{e}_i \sim D_{\mathbf{c}_i+\mathcal{L}}$ conditioned on $\mathbf{c}_i$.

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Soundness

- If $\lambda_1(\mathcal{L}^*) \le 1/10$, only $2^{-\Omega(n)}$-fraction of $\{\mathbf{c}_i\}$ have valid proof $\{\mathbf{e}_i\}$.

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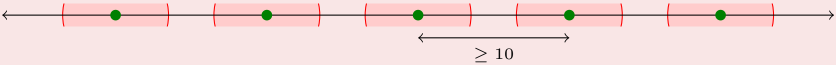
  Intuition: projecting $\mathcal{L}$ and sufficiently small $\mathbf{e}_i$ onto span$(\mathbf{v}^*)$ yields



Unlikely that all the random $\mathbf{c}_i$ project to 'good' region.

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Conclusion

Completeness, simulation (for $\eta \leq 1 \Longleftarrow \lambda_1^* > \sqrt{n}$)

& soundness (for $\lambda_1^* \leq 1/10$)

$\Downarrow$

this is a NISZK for $O(\sqrt{n})$-coGapSVP.

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- Random String: uniform cosets $\mathbf{c}_i \leftarrow \mathbb{R}^n/\mathcal{L}$ for $i = 1, \ldots, m$.
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- Verifier: accept iff each $\mathbf{e}_i \in \mathbf{c}_i + \mathcal{L}$ and $\sigma_1(\sum \mathbf{e}_i \mathbf{e}_i^T) \leq 3m$.
- Simulator: first sample $\mathbf{e}_i$ from continuous Gaussian as proof, then output cosets $\mathbf{c}_i = \mathbf{e}_i + \mathcal{L}$ as random string.

Conclusion

Completeness, simulation (for $\eta \leq 1 \Longleftarrow \lambda_1^* > \sqrt{n}$)

& soundness (for $\lambda_1^* \leq 1/10$)

$$\Downarrow$$

this is a NISZK for $O(\sqrt{n})$-coGapSVP.

- Can the same proof system work for GapSPP?

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- Conclusion: $\approx \log n$ gap in $\eta(\mathcal{L})$ between completeness, soundness.

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- Yields a Karp reduction γ-GapSPP$_\varepsilon \leq$ EntropyApproximation, with $\gamma = O(\log(n)\sqrt{\log(1/\varepsilon)})$ for any $\varepsilon \in (0, 1/2)$.

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Thanks!