Limits on the Hardness of Lattice Problems in ℓ_p Norms

Chris Peikert

SRI International

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Lattices and Their Problems

Let $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{R}^n$ be linearly independent.

The n-dim lattice $\mathcal{L}$ having basis $\mathbf{B}$ is:

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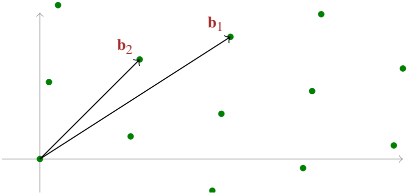

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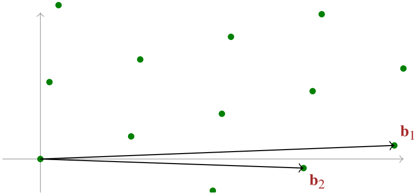

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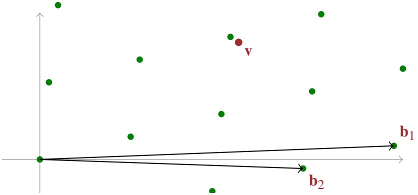

Close Vector Problem (CVP$_\gamma$)

Approximation factor $\gamma = \gamma(n)$, in some norm $\|\cdot\|$.

- Given basis $\mathbf{B}$ and point $\mathbf{v} \in \mathbb{R}^n$, distinguish $\text{dist}(\mathbf{v}, \mathcal{L}) \leq 1$ from $\text{dist}(\mathbf{v}, \mathcal{L}) > \gamma$ (otherwise, don't care.)

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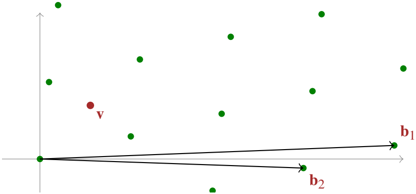

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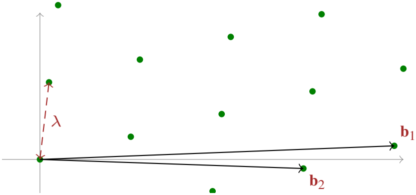

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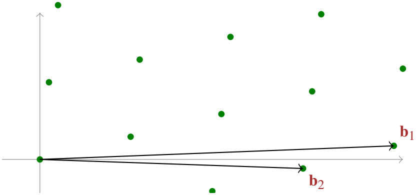

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Usually use ℓ_p norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

Algorithms and Hardness

Algorithms for SVP$_\gamma$ & CVP$_\gamma$

- $\gamma(n) \sim 2^n$ approximation in poly-time [LLL,Babai,Schnorr]
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NP-Hardness **(some randomized reductions...)**

- In any ℓ_p norm, SVP$_\gamma$ hard for any $\gamma(n) = O(1)$ [Ajt,Micc,Khot,ReRo]
- In any ℓ_p norm, CVP$_\gamma$ hard for any $\gamma(n) = n^{O(1/\log\log n)}$ [DKRS,Dinur]
- Many other problems (CVPP, SIVP) hard as well ...

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(Can generalize to ℓ_p norms, but lose up to $\sqrt{n}$ factors.)

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Techniques

- New analysis of prior algorithms [AharRegev,MiccRegev,Regev,...]
- General analysis of discrete Gaussians over lattices
- Introduce ideas from [Ban95] to complexity

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A Bit Odd

- Can't show anything new for $1 \leq p < 2$...

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3. What's going on with $p < 2$?

   (Beating $n^{1/p}$ for even a single p has implications for codes.)
4. Are all ℓ_p norms ($p \geq 2$) equivalent?

Gauss meets Lattices

Define Gaussian function $\rho(\mathbf{x}) = \exp(-\pi \|\mathbf{x}\|_2^2)$ over $\mathbb{R}^n$.

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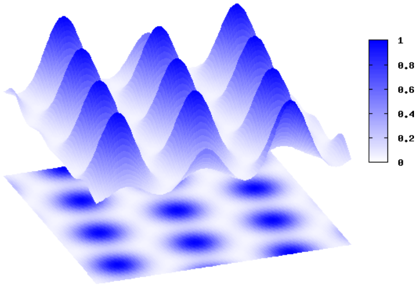

Define

$$f(\mathbf{x}) = \frac{\sum_{\mathbf{v}\in\mathcal{L}} \rho(\mathbf{x}-\mathbf{v})}{\sum_{\mathbf{v}\in\mathcal{L}} \rho(\mathbf{v})} = \frac{\rho(\mathcal{L}-\mathbf{x})}{\rho(\mathcal{L})}.$$

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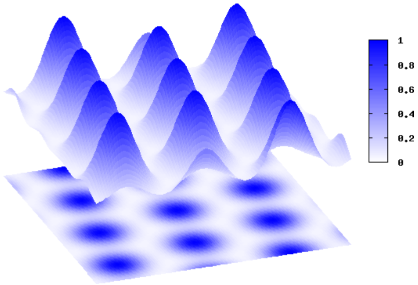

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Properties of f

- If $\mathrm{dist}_2(\mathbf{x}, \mathcal{L}) \le \frac{1}{10}$, then $f(\mathbf{x}) \ge \frac{1}{2}$. (Easy.)
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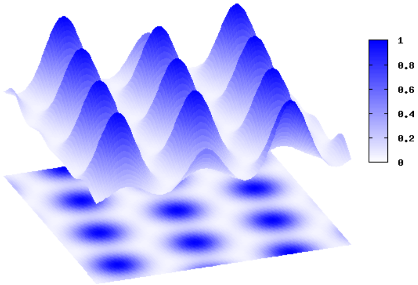

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Enter Aharonov & Regev…

- A compact & verifiable representation of $f \Rightarrow \mathrm{CVP}_{10\sqrt{n}} \in \mathrm{coNP}$.

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Lemma [Ban93]

For any lattice $\mathcal{L}$ and $\mathbf{x} \in \mathbb{R}^n$,

$$\frac{\rho((\mathcal{L} - \mathbf{x}) \setminus \sqrt{n} \cdot \mathcal{B}_2)}{\rho(\mathcal{L})} < 2^{-n}.$$

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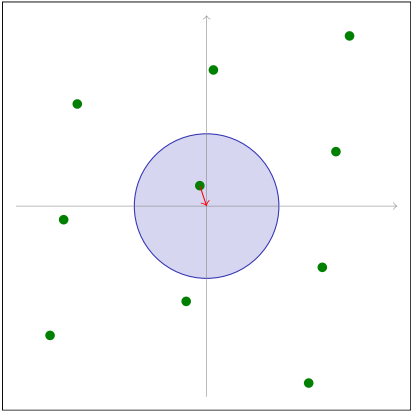

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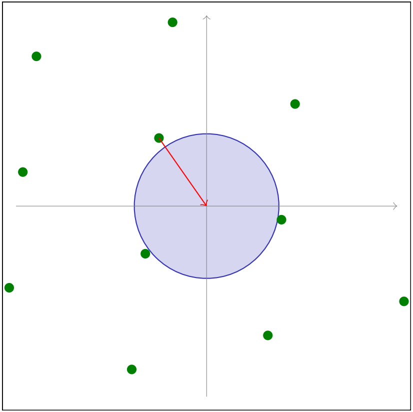

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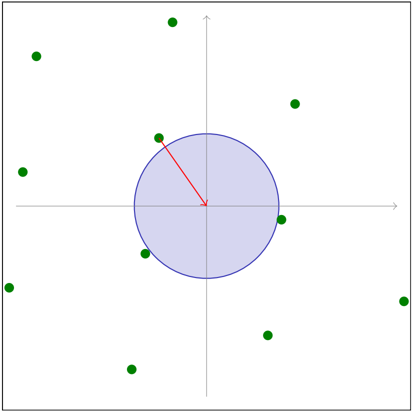

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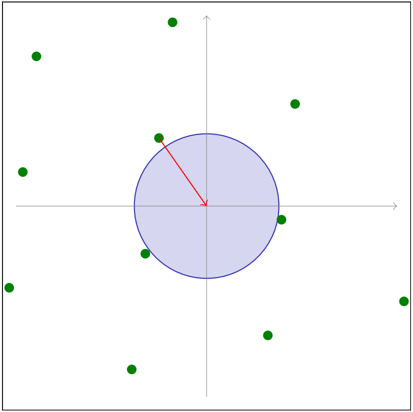

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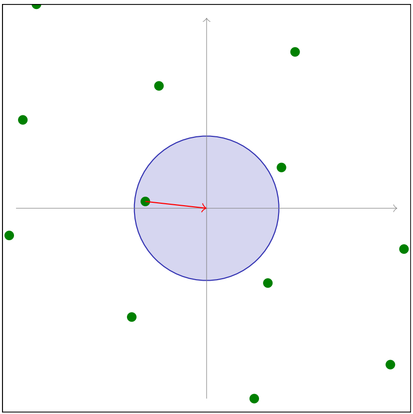

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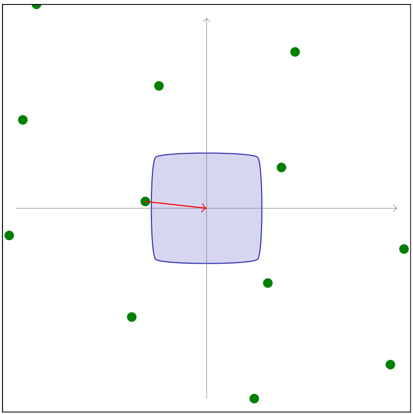

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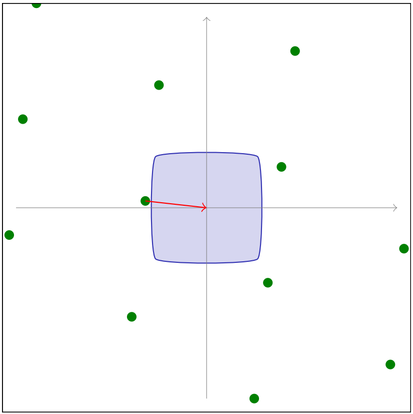

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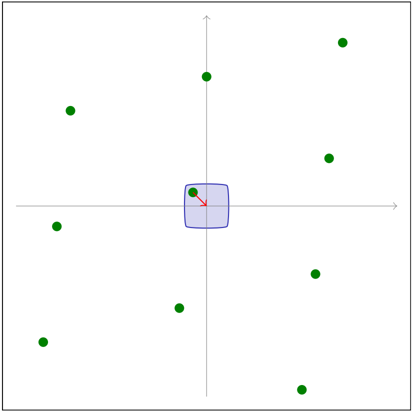

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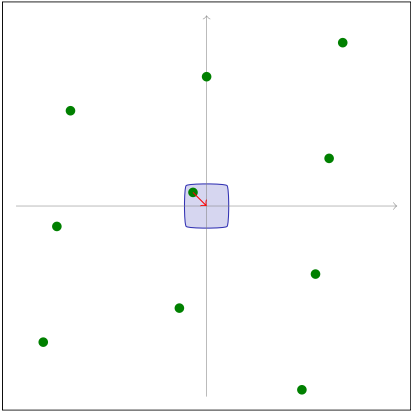

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- Therefore in ℓ_p norm, $\mathrm{CVP}_{10 c_p \sqrt{n}} \in \mathrm{coNP}$.

Generalizing to ℓ_p Norms

Lemma [Ban95]

For any $p \in [1, \infty)$, there exists a constant c_p:

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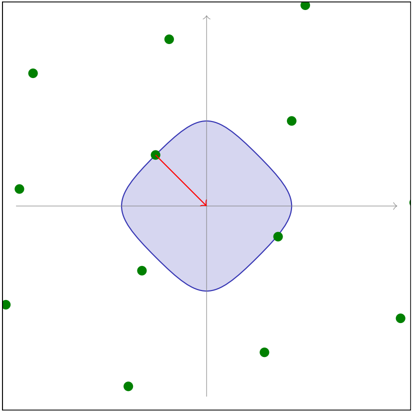

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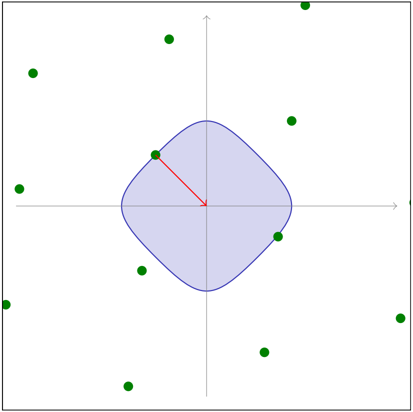

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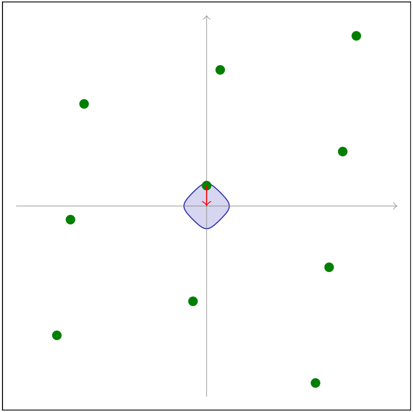

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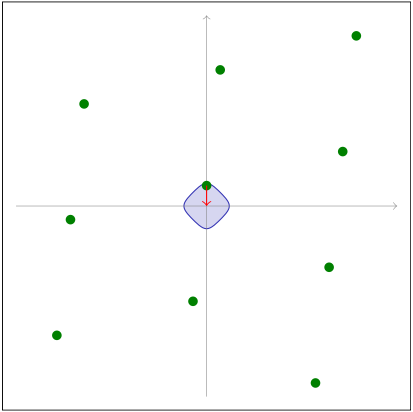

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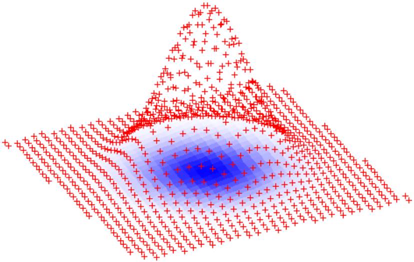

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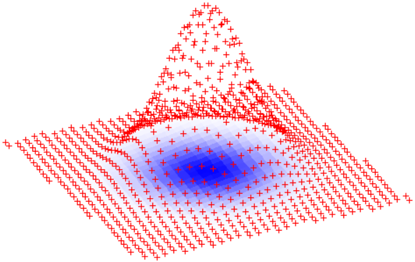

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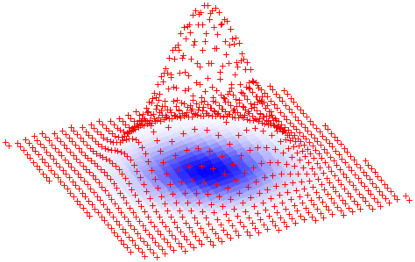

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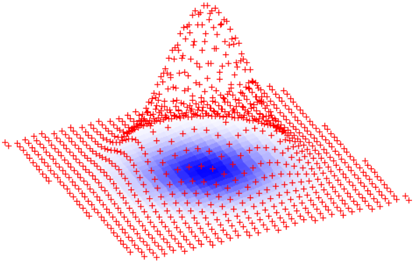

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A: Just like those from a continuous Gaussian!

$$\mathop{\mathrm{E}}_{\mathbf{x} \sim D_{\mathcal{L}}} \left[\|\mathbf{x}\|_p \right] \approx \sqrt{p} \cdot n^{1/p}$$

Proof Highlights

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For any $r \geq 0$,

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Jensen & Linearity

$$\mathop{\mathrm{E}}_{\mathbf{x}\sim D_{\mathcal{L}}}\left[\|\mathbf{x}\|_p\right] \;\leq\; \left(\mathrm{E}\left[\|\mathbf{x}\|_p^p\right]\right)^{1/p} \;=\; (n \cdot \mathrm{E}[|x_i|^p])^{1/p} \;\leq\; \sqrt{p} \cdot n^{1/p}.$$

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3. We should pay more attention to the ℓ_1 norm.