Pseudorandomness of Ring-LWE for Any Ring and Modulus

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(to appear, STOC'17)

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Lattice-Based Cryptography

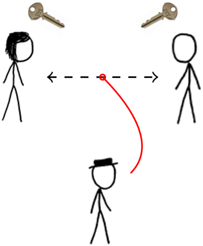
$y = g^x \bmod p$

$N = p \cdot q$

$m^e \bmod N$

$e(g^a, g^b)$

$\Longrightarrow$



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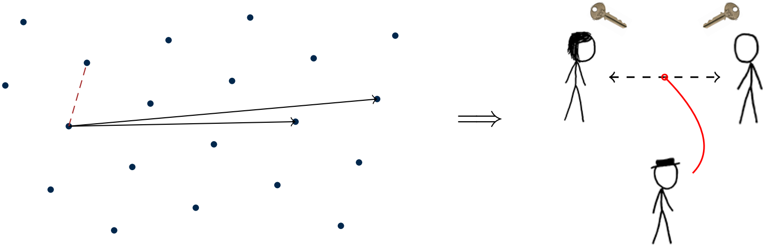

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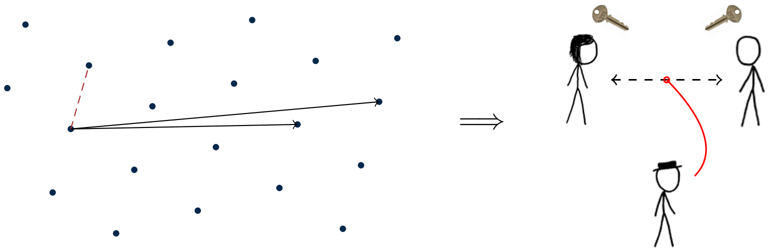

Main Attractions

- Efficient: linear, embarrassingly parallel operations

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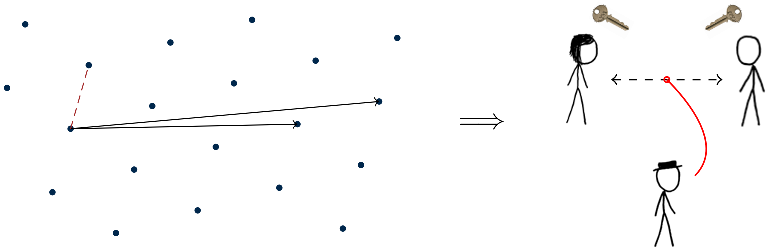

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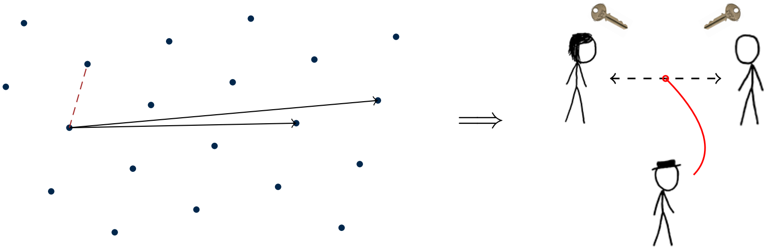

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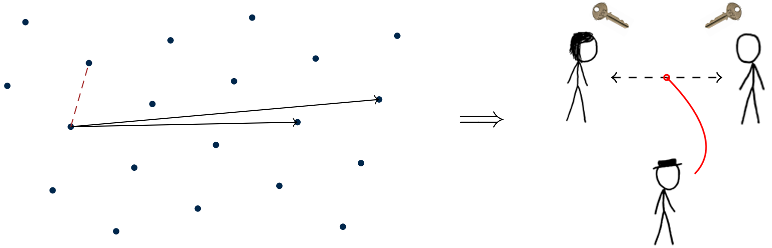

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- Efficient: linear, embarrassingly parallel operations
- Resists quantum attacks (so far)
- Security from worst-case assumptions
- Solutions to 'holy grail' problems in crypto: FHE and related

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Learning With Errors [Regev'05]

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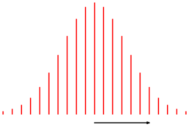
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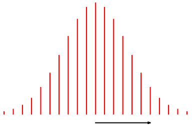
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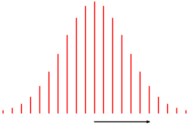
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LWE is Hard and Versatile

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(n/α)-SIVP on n-dim lattices $\leq$ search-LWE $\leq$ decision-LWE $\leq$ much crypto

(quantum [R'05]) [BFKL'93,R'05,...]

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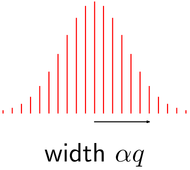

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- *Classically*, GapSVP $\leq$ search-LWE (worse params) [P'09,BLPRS'13]

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- Increasing q, α yields a weaker ultimate hardness guarantee.

LWE is Efficient (Sort Of)

$$\left(\cdots\ \mathbf{a}_i\ \cdots\right)\begin{pmatrix}\vdots\\ \mathbf{s}\\ \vdots\end{pmatrix} + e \ = \ b \in \mathbb{Z}_q$$

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- Cryptosystems have rather large keys: $\Omega(n^2 \log^2 q)$ bits:

$$pk = \underbrace{\begin{pmatrix} \quad \vdots \quad \\ \mathbf{A} \\ \vdots \end{pmatrix}}_{n} \quad , \quad \left.\begin{pmatrix} \vdots \\ \mathbf{b} \\ \vdots \end{pmatrix}\right\} \Omega(n)$$

Wishful Thinking…

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- Same ring structures used in NTRU cryptosystem [HPS'98],

  & in compact one-way / CR hash functions [Mic'02,PR'06,LM'06,. . .]

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Search: find secret ring element $s \in R_q^\vee$, given independent samples

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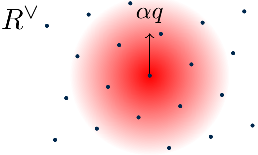

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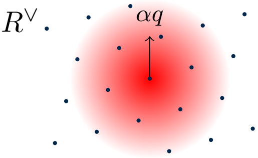

Decision: distinguish $(a_i\ ,\ b_i)$ from uniform $(a_i\ ,\ b_i) \in R_q \times R_q^\vee$

Hardness of Ring-LWE [LPR'10]

worst-case (n^c/α)-SIVP
on *ideal* lattices in R $\le$ search R-LWE$_{q,\alpha}$ $\le$ decision R-LWE$_{q,\alpha}$

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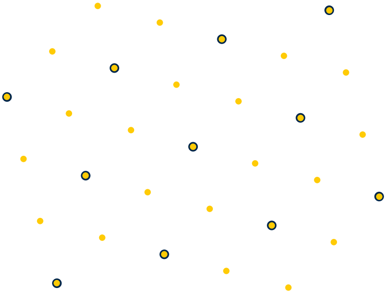
(Ideal $\mathcal{I} \subseteq R$: additive subgroup, $x \cdot r \in \mathcal{I}$ for all $x \in \mathcal{I}, r \in R$.)



$R = \mathbb{Z}[X]/(1 + X + X^2)$

ideal $\mathcal{I} = 3R + (1 - X)R \subset R$

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Hardness of Ring-LWE [LPR'10]

worst-case (n^c/α)-SIVP on *ideal* lattices in R $\underset{\nwarrow}{\leq}$ search R-LWE$_{q,\alpha}$ $\underset{\nwarrow}{\leq}$ decision R-LWE$_{q,\alpha}$

(quantum, *any* $R = \mathcal{O}_K$)

(classical, any *Galois* R)

Large disparity in known hardness of search versus decision:

Search: any number ring, any $q \geq n^c/\alpha$.

Decision: any Galois number ring (e.g., cyclotomic),
any highly splitting prime $q = \text{poly}(n)$.

Can then get any q by mod-switching, but increases α [LS'15]

- Decision has no known worst-case hardness in non-Galois rings.
- But no examples of easy(er) decision when search is worst-case hard!

Our Results

Main Theorem: Ring-LWE is Pseudorandom in Any Ring

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- Seems to adapt to ‘module’ lattices/LWE w/techniques from [LS’15]

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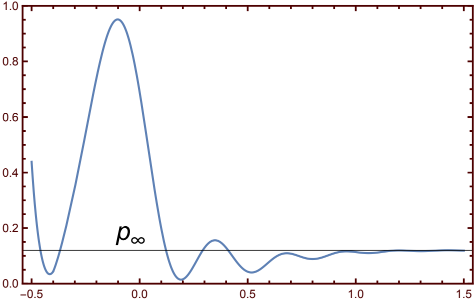

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③ $p(\log \alpha) - p(\infty)$ is noticeable, so there is a noticeable change in p somewhere between $\log \alpha$ and $\log n$.

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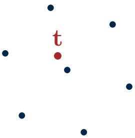
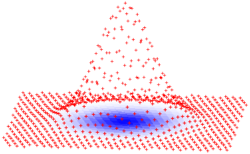


BDD$_{\mathcal{L}^*}$, dist d

$+$



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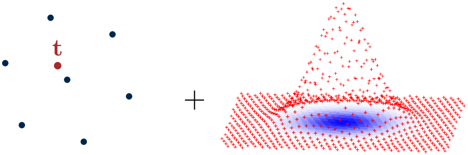

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($D_{\mathcal{L},r}$ samples come from previous iteration, quantumly. They're eventually narrow enough to solve SIVP on $\mathcal{L}$.)

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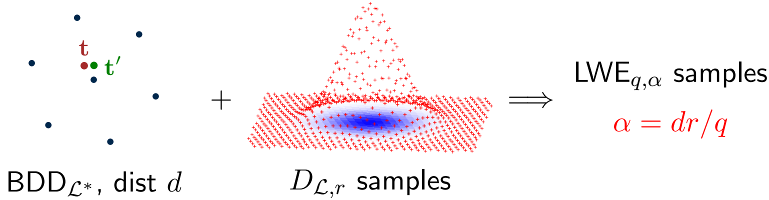

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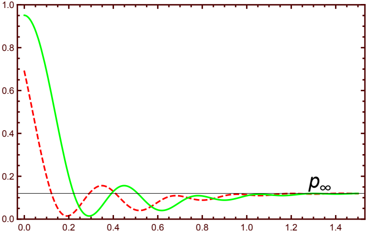
- Idea: perturb $\mathbf{t}$, use $\mathcal{O}$ to check whether we're closer to $\mathcal{L}^*$ by how $\alpha = dr/q$ changes.

  We get a 'suffix' of $p(\cdot)$.



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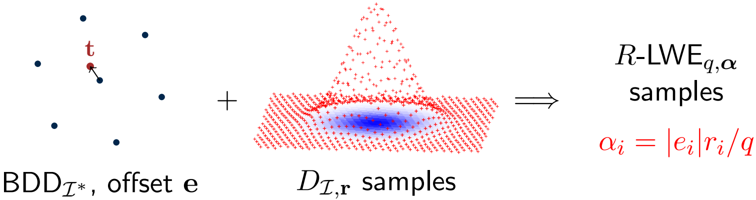

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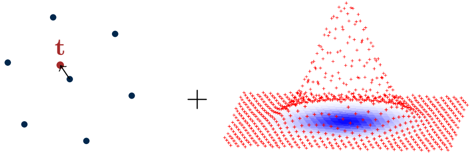

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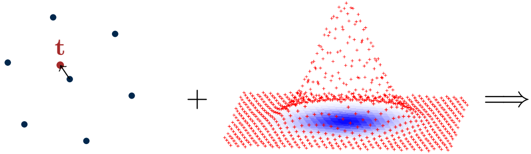

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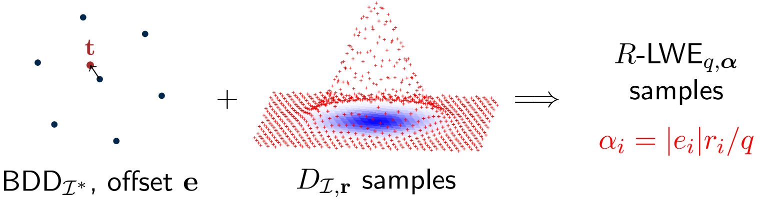

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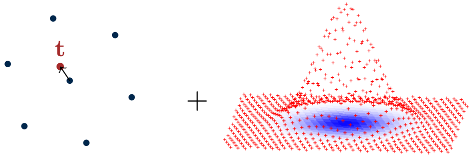

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