Location of Non-Zeros in Sparse Solutions of Underdetermined Linear Systems of Equations

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Abstract— **The problem of computing sparse (mostly zero) or sparsifiable (by linear transformation) solutions to greatly underdetermined linear systems of equations has applications in compressed sensing. The locations of the nonzero elements in the solution is a much smaller set of variables than the solution itself. We present explicit equations for the relatively few variables that determine these nonzero locations and also propose an iterative algorithm for their solution.**

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I. INTRODUCTION

A. Problem Statement

The goal is to solve underdetermined linear system

$$y = \tilde{H}\tilde{z} = \tilde{H}W'W\tilde{z} = G\tilde{x}; G = \tilde{H}W'; \tilde{x} = W\tilde{z} \quad (1)$$

- G and $\tilde{H}$ are both M×N matrices, with N>>M;
- W is an N×N unitary transformation matrix that sparsifies the unknown column N-vector $\tilde{z}$ to $\tilde{x}$;
- $\tilde{x}$ has only K out of N non-zero elements; K<M;
- $y, \tilde{H}, W, G, M, N$ are known; $\tilde{x}, \tilde{z}$ are unknown.

The sparsifying transformation W is any orthonormal transform, such as the wavelet transform.

We note that if G is substantially underdetermined $N >> M$, but each row is known to be bandlimited to $\pi\frac{M}{N}$, then our non-iterative algorithm [1] can be used. We have also used related but different approaches to X-ray crystallography [2] and to the limited-angle tomography formulation of SAR [3].

B. Relevant Previous Approaches

Use of sparseness as side information in signal reconstruction goes back at least as far as 1979 [4]. Most early work considered the deconvolution of sparse 1D spike trains arising in reflections from layered media in seismic exploration, although [5] considered 1D reconstruction from bandlimited data. The first use of sparseness in 2D (image) reconstruction of which we are aware is [6]. All of this early work minimized the ℓ_1 norm (sum of absolute values) of the signal, using linear programming. The idea was that the ℓ_1 norm solution lies on a vertex of the simplex and so is sparse. This idea has recently been put on a firmer theoretical ground in [7] and other recent papers. More recent work using this approach is [8]-[10]. Using the ℓ_1 norm for the signal and ℓ_2 for the error is called LASSO. The problem with this approach is the amount of computation required by linear programming for image reconstruction.

Another way of formulating the linear sparse reconstruction problem is as a matrix "subset selection" problem [11]-[14]. The forward greedy algorithm successively selects the matrix column closest (in the mean-square sense) to the residual error resulting from the previous matrix column selections. The backward greedy algorithm starts with a general solution and successively removes the matrix column that increases the mean-square residual error the least. The latter algorithm has been shown to give the correct answer if the noise level is sufficiently small [13]. However, again the problem is the amount of computation required here by subset selection.

Still another recent approach is to include a thresholding constraint in a Landweber-like iterative algorithm [15]-[18], often arising from statistical image priors that implicitly (but not explicitly) maximize sparsity. This is a straightforward approach, and unlike the above methods requires reasonable computation for 2-D problems. However, even in the absence of noise, convergence to an optimally-sparse solution is not in general guaranteed for these algorithms.

C. New Approach of This Paper

By focusing only on the K *locations* of the nonzero elements of the solution, we greatly reduce the size of the problem from N to K, where N>>M>K. Furthermore, by avoiding use of the ℓ_1 norm to sparsify the solution, we avoid all of the requirements of that approach (random selection of matrices, M> $K \log K$).

II. Derivation of Equations

A. Fourier Reformulation

Recall from (1) that the goal is to solve $y = G\tilde{x}$, where $\tilde{x}$ has K nonzero elements. Rewrite (1) as

$$y = G\tilde{x} \rightarrow 0 = [-y|G][1|\tilde{x}]^T = Hx \quad (2)$$

where we have defined $H = [-y|G]$ and x has K+1 nonzero elements, including a 1^{st} element of one.

The N-point discrete Fourier transform (DFT) is

$$\hat{X} = Dx, \quad [D]_{nk} = e^{-j2\pi(n-1)(k-1)/N}. \tag{3}$$

Let $\hat{H}$ be the matrix of the complex conjugates of the N-point DFTs of each row of H:

$$\hat{H} = HD^H, [\hat{H}]_{ik} = \sum_{n=1}^{N} [H]_{in} e^{j2\pi(n-1)(k-1)/N}. \tag{4}$$

Then Parseval's theorem leads to

$$0 = NHx = H(D^H D)x = (HD^H)(Dx) = \hat{H}\hat{X}. \tag{5}$$

B. Indicator Function

Let $\{n_i, 1 \le i \le K+1\}$ be the indices of the nonzero elements of x, so that only $x_{n_i} \neq 0$. Let

$$s_n = \prod_{i=1}^{K+1} (e^{j2\pi n/N} - e^{j2\pi n_i/N}). \tag{6}$$

- $s_{n_i} = 0$ and $s_n \neq 0$ for $n \neq n_i$;
- The DFT $\hat{S}$=Ds has length K+2;
- These K+2 numbers indicate nonzero x_n.

Now let $[\hat{S}]$ and $[\hat{X}]$ be the circulant matrices having first rows $\hat{S}$ and $\hat{X}$ and eigenvalues s_n and x_n. Since $[\hat{S}]$ and $[\hat{X}]$ have the same matrix of eigenvectors D, and $s_n x_n$=0, we have

$$[\hat{S}][\hat{X}] = [0] \rightarrow [\hat{S}]\hat{X} = 0 \tag{7}$$

Or, the DFT of $s_n x_n$=0 is $\hat{S} \copyright \hat{X}$=0, where © denotes circular convolution. This can be written in matrix form as (7). Also note that $\sum \hat{S}_k = 0$ since s_1=0 (the 1^{st} element of x_n is one), so $\hat{S}_k$ has K+1 unknowns (K and an overall scale factor).

C. Reduced-Rank Matrix Equation

Combining (5) and (7) into a single equation gives

$$\begin{bmatrix} \hat{H} \\ [\hat{S}] \end{bmatrix} \hat{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

This is M+N equations in N+K+1 unknowns, which are overdetermined if K≤M-1 since an overall scale factor cancels. Looking only at the matrix, the condition that it drops rank gives M+1 equations in K+1 unknowns. This is overdetermined if K<M, and the number of unknowns is K+1<<N+K+1.

These equations can be solved using any of these:

- Structured total least squares, in which the K+1 unknowns are perturbed in the direction that reduces the minimum singular value toward zero;
- Multiplying out the minors to give an explicit system of simultaneous polynomial equations;
- An iterative algorithm in which the steps are:

1. Compute the SVD and determine the minimum Frobenius norm perturbation that drops its rank;
2. Insert altered $\hat{S}_k$ in the original matrix;
3. Repeat. Each operation is norm-reducing; the global convergence theorem guarantees convergence.

Our numerical experiments indicate that this converges to the correct solution if the initial $\hat{S}_k$ are close to the actual $\hat{S}_k$. The huge SVD computation makes another algorithm desirable. This is discussed next.

D. Simultaneous Quadratic Equations

Multiplying each of (5) by $\hat{S}_k$ and augmenting this to (7) gives the simultaneous quadratic equations

$$\begin{bmatrix} \hat{H} & 0 & \cdots & 0 \\ 0 & \hat{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{H} \\ C^0 & C^1 & \cdots & C^K \end{bmatrix} \begin{bmatrix} \hat{S}_{K+2}\hat{X}_0 \\ \vdots \\ \hat{S}_{K+2}\hat{X}_N \\ \vdots \\ \hat{S}_0\hat{X}_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

where the N×N circulant shift matrix C has the form

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{10}$$

a set of simultaneous quadratic equations in $\hat{S}_i\hat{X}_j$.

We propose the following algorithm for solving (9):

1. Start with initial $\hat{X}$ from the least-squares solution to (1). Compute $\hat{S}$ from the smallest values of x;
2. Form the N(K+2)-vector $\hat{S}_i\hat{X}_j$;
3. Project $\hat{S}_i\hat{X}_j$ onto the nullspace of (9);
4. Arrange N(K+2)-vector to an N×(K+2) matrix;
5. Compute the minimum Frobenius norm perturbation of this matrix that drops its rank to one;
6. Use the result to define new $\hat{X}$ and $\hat{S}$;
7. Repeat. Each operation is norm-reducing; the global convergence theorem guarantees convergence.

Our numerical experiments indicate that this converges to the correct solution if the initial $\hat{X}_k$ are close to the actual $\hat{X}_k$. The difference is that the rank-one approximation can be computed quickly using the power method; no SVD is required.

III. Numerical Examples

A. Tiny Example

This merely illustrates the forms of the equations. The goal is to solve the underdetermined system

$$\begin{bmatrix} 5 \\ 16 \\ 38 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 5 & 9 \\ 2 & 6 & 5 & 3 & 5 \\ 8 & 9 & 7 & 9 & 3 \end{bmatrix} \tilde{x} \tag{11}$$

with the side information that only two elements, locations unknown, of $\tilde{x}$ are nonzero.

The problem is reformulated as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 4 & 1 & 5 & 9 \\ -16 & 2 & 6 & 5 & 3 & 5 \\ -38 & 8 & 9 & 7 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{12}$$

with x_1=1. Combining (5) and (7) gives

$$\begin{bmatrix} 15 & 9.54e^{-j2.19} & 14.8e^{-j2.72} & -7 & 14.8e^{j2.72} & 9.54e^{j2.19} \\ 5 & -22 & 19.7e^{-j2.87} & -19 & 19.7e^{j2.87} & -22 \\ -2 & 48.7e^{j3.05} & 45.7e^{j3.05} & -38 & 45.7e^{-j3.05} & 48.7e^{-j3.05} \\ \hat{S}_4 & \hat{S}_3 & \hat{S}_2 & \hat{S}_1 & 0 & 0 \\ 0 & \hat{S}_4 & \hat{S}_3 & \hat{S}_2 & \hat{S}_1 & 0 \\ 0 & 0 & \hat{S}_4 & \hat{S}_3 & \hat{S}_2 & \hat{S}_1 \\ \hat{S}_1 & 0 & 0 & \hat{S}_4 & \hat{S}_3 & \hat{S}_2 \\ \hat{S}_2 & \hat{S}_1 & 0 & 0 & \hat{S}_4 & \hat{S}_3 \\ \hat{S}_3 & \hat{S}_2 & \hat{S}_1 & 0 & 0 & \hat{S}_4 \end{bmatrix}$$

This 9×6 matrix must have rank=5. Hence four minors are zero, and these are four equations in the four unknowns $\hat{S}$. However, since the sum of the elements of $\hat{S}$ are zero, there are really only three unknowns. Solution of this overdetermined system of equations

$$\hat{S} = [3 + j5.2, -6, -3 - j5.2, 6, 0, 0] \tag{13}$$

and an inverse DFT of this yields

$$s = [0, 0, 1.73e^{j0.52}, 0, 3.46e^{j0.52}, 3e^{j2.1}] \tag{14}$$

which identifies $x_1 = 1, x_2, x_4$ as nonzero x_n. The values of these nonzero x_n are easily found to be

$$x = [1, 3, 0, 2, 0, 0]' \tag{15}$$

Matlab code for this example:

```
clear;X=[3 0 2 0 0]'; %2-Sparse solution
H=[1 4 1 5 9;2 6 5 3 5;8 9 7 9 3]; %from pi
Y=H*X; FH=(fft([-Y H]'))'; FX=fft([1;X]);
S= (exp(j*2*pi*[0:5]/6)-exp(j*2*pi*0/6));
S=S.*(exp(j*2*pi*[0:5]/6)-exp(j*2*pi*1/6));
S=S.*(exp(j*2*pi*[0:5]/6)-exp(j*2*pi*3/6));
FS=fft(S); %Indicator function S.*[1;X]'=0
C=[FS(4) 0 0 FS(1:3)];R=[FS(4:-1:1) 0 0];
T=toeplitz(C,R);[FH;T]*FX %=0 to roundoff
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