Error Exponent Region for Gaussian Broadcast Channels

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***Abstract*— In this work, we introduce the notion of the error exponent region for a multi-user channel. This region specifies the set of error-exponent vectors that are simultaneously achievable by all users in the multi-user channel. This is done by associating different probabilities of error for different users, contrary to the traditional approach where a single *probability of system error* is considered. We derive an inner bound (achievable region) and an outer bound for the error exponent region of a Gaussian broadcast channel.**

I. Introduction

It is a well-known fact that the error exponent for a single-user channel provides the rate of exponential decay of the average probability of error as a function of the block length of the codebooks [1], [2]. Conceptually, in a single-user channel, the error exponent is a function of the operating point R and the channel capacity C and in particular, a non-decreasing function of the difference between R and C. Therefore, there is a tradeoff between the rate and the error exponent in a single-user channel. One can increase the error exponent by reducing the rate. The concept of the error exponent was extended to a Gaussian multiple access channel (MAC) in [3], [4], where an upper bound on the *probability of system error* (i.e., the probability that any user is in error) was derived for random codes.

In many applications of wireless networks, different users might have different reliability requirements. For instance, in an uplink (or downlink) of a cellular system, a user running an FTP application might have more stringent reliability requirements than a user running a multimeadia application which is designed for graceful degradation. Based on the traditional approaches [3], [4] which consider a single *probability of system error*, a network can only be designed to satisfy the most stringent reliability requirement. This might result in a mismatch of resources allocation, and thus, it is inherently suboptimal.

Motivated by the above observation, in this work, we consider a new approach in analyzing the users' performance in a multi-user scenario. In addition to the rate vs. performance tradeoffs that exist in traditional approaches, our approach realizes new degrees of freedom that enable a richer tradeoff among users' performance. Our approach hinges on the following two observations.

First, one can define a probability of error for each user, which, in general may be different for different users. Therefore, there are multiple error exponents, one for each user, for a given multi-user channel.

Second, in contrast to a single-user channel where the error exponent is fixed for a given rate, in a multi-user channel one can tradeoff the error exponents among different users even for fixed rates. To illustrate this novel point, consider the capacity region of a two-user broadcast channel as shown in Fig. 1(b). As expected, the error exponents for the two users are functions of both the operating point A and the channel capacity. However, unlike the case in a single-user channel where the channel capacity boundary is a single point, in a multi-user channel we have multiple points on the capacity boundary (e.g. B, D in Fig. 1(b)). Thus it is expected that one can get different error exponents depending on which particular point on the capacity boundary is considered. Furthermore, it might be possible to trade off error exponents between users by considering different points on the capacity boundary. For instance, consider an operating point A (corresponding to a rate pair (R_1, R_2)) with respect to a boundary point B in Fig. 1(b). It is intuitive to expect that the error exponent for user 1 is smaller than that of user 2, since user 1 operates at rate R_1 which is very close to his capacity (determined by B), while user 2 backs off significantly from his capacity (determined again by B). On the other hand, if we consider point A with respect to the boundary point D, we then expect the error exponent for user 1 to be larger than that of user 2. Therefore, a tradeoff of error exponents between users might be possible by considering different points on the capacity boundary. It is our intention in this paper to formalize these ideas by showing that such tradeoff indeed exists and by proposing constructive strategies to achieve it.

Before continuing, we introduce the notion of error exponent region (EER). For a given operating point, the error exponent region consists of all achievable error exponents when the channel is operated at that point. For example, the error exponent region for a single-user channel operated at rate R is a line segment from the origin to the error exponent $E(R)$ (see Fig. 2(a)). For a broadcast channel operated at point A (see Fig. 1(b)), the error exponent region is a two-dimensional region which depends on rates R_1 and R_2 (see Fig. 2(b)). The concept of the error exponent region is very similar to the concept of the channel capacity region (CCR). In the EER, it is possible to increase user 1's error exponent by decreasing user 2's error exponent. This is similar to the idea of increasing the data rate of user 1 by reducing the data rate of user 2 in the CCR. However, there is a fundamental difference between CCR and EER. For a given channel, there is only one CCR. One the other hand, an EER depends on the channel operating point, and for a given channel, there are numerous EERs depending on which operating point we consider. Therefore, when we refer to an EER, we need to specify the channel operating point.

The rest of the paper is structured as follows. In Section II, we derive the achievable error exponent region by superposition and the achievable error exponent region by time-sharing in a Gaussian broadcast channel. The union of these two regions is an inner bound for the error exponent region. In Section III, we use a different decoding scheme to improve the error exponent region derived by superposition in Section II. In Section IV, we derive outer bounds for the error exponent regions of a discrete memoryless broadcast channel (DMBC) and the error exponent region of a Gaussian broadcast channel. We conclude our work in Section V. The existence of a good codebook which achieves the average error exponents of random codebooks using superposition encoding is proved in Appendix.

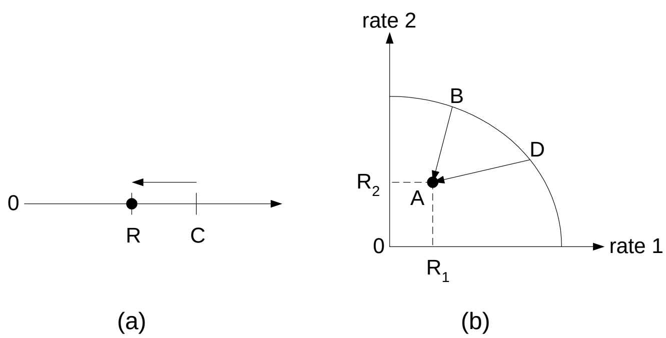


Fig. 1. Capacity region for (a) single-user, and (b) broadcast channels.

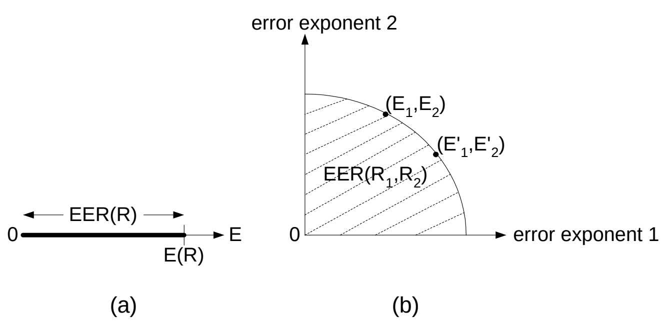


Fig. 2. (a) The error exponent region associated with rate R (single-user). (b) The error exponent region associated with operating point (R_1, R_2).

II. Achievable Error Exponent Region for Gaussian Broadcast Channels

Consider a two-user Gaussian broadcast channel

$$Y_1 = X + Z_1 \tag{1}$$
$$Y_2 = X + Z_2, \tag{2}$$

where X is the channel input with power constraint P, and Y_1 and Y_2 are the channel outputs for user 1 and user 2. Assume that the noise power for Z_1 is σ_1^2 and the noise power for Z_2 is σ_2^2. The capacity boundary for a Gaussian broadcast channel is achieved by different input distributions. In Fig. 1(b), the operating point B is achieved by $X = X_1 + X_2$ with Gaussian distributions $\mathcal{N}(0, \alpha_1 P)$ and $\mathcal{N}(0, (1-\alpha_1)P)$ for X_1, X_2, respectively, but the point D is achieved by another pair of Gaussian distributions $\mathcal{N}(0, \alpha_2 P)$ and $\mathcal{N}(0, (1-\alpha_2)P)$ ($0 < \alpha_1 < \alpha_2 < 1$). Therefore, we expect the error exponents for the operating point A evaluated with respect to B to be different from those evaluated with respect to D. In the receivers side, we decode users' messages using *joint* maximum likelihood (ML) decoding, i.e., decoding user 1's message based on the pair (i,j) maximizing $P(Y_1^N | X_{1i}^N, X_{2j}^N)$ and decoding user 2's message based on the (i,j) maximizing $P(Y_2^N | X_{1i}^N, X_{2j}^N)$, where X_{1i}^N, X_{2j}^N, Y_1^N, and Y_2^N are the transmitted codewords and the received data for user 1 and user 2 with block length N, respectively. Based on this assumptions, we derive achievable error exponents for user 1 and user 2 in a Gaussian broadcast channel as

$$E_1^s = \min\{E(R_1, \frac{\alpha P}{\sigma_1^2}), E_{t3}(R_1 + R_2, \frac{\alpha P}{\sigma_1^2}, \frac{(1-\alpha)P}{\sigma_1^2})\} \tag{3}$$

$$E_2^s = \min\{E(R_2, \frac{(1-\alpha)P}{\sigma_2^2}), E_{t3}(R_1 + R_2, \frac{\alpha P}{\sigma_2^2}, \frac{(1-\alpha)P}{\sigma_2^2})\}, \tag{4}$$

where the superscript "s" denotes superposition, and $0 < \alpha < 1$. In (3), (4), $E(R, SNR)$ is the maximum of the single-user random coding exponent and the single-user expurgated exponent [1], [2], and $E_{t3}(R_1 + R_2, SNR_1, SNR_2)$ is the random coding exponent for the type 3 error in a two-user Gaussian multiple access channel [3]. An explicit expression for E_{t3} is

$$E_{t3}(R_3, SNR_1, SNR_2) = \max_{\rho, \theta_1, \theta_2} \{E_{t3,0}(\rho, \theta_1, \theta_2) - \rho R_3\} \tag{5}$$

$$E_{t3,0}(\rho, \theta_1, \theta_2) = (1+\rho) \ln \left[\frac{e\sqrt{\theta_1 \theta_2}}{1+\rho} \right] - \frac{\theta_1 + \theta_2}{2} + \frac{\rho}{2} \ln \left[1 + \frac{SNR_1}{\theta_1} + \frac{SNR_2}{\theta_2} \right], \tag{6}$$

where the maximization is over $0 \le \rho \le 1$, and $0 < \theta_1, \theta_2 \le 1 + \rho$.

In Fig. 3(a), the solid curve is the boundary of the achievable EER obtained by superposition. In the following, we propose a simple scheme (time-sharing) to enlarge the achievable EER beyond the achievable region by superposition. The achievable error exponents for user 1 and user 2 by time-sharing are

$$E_1^{ts} = \alpha E\left(\frac{R_1}{\alpha}, \frac{P}{\sigma_1^2}\right) \quad (7)$$

$$E_2^{ts} = (1-\alpha) E\left(\frac{R_2}{1-\alpha}, \frac{P}{\sigma_2^2}\right), \quad (8)$$

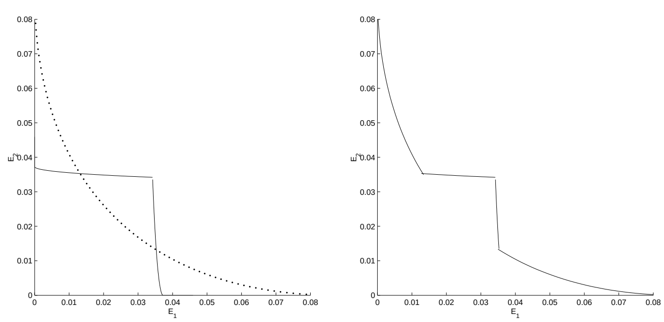
where the superscript "ts" denotes time-sharing, and $0 < \alpha < 1$. In Fig. 3(a), the dotted curve is the achievable EER by time-sharing. The union of the superposition achievable EER and the time-sharing achievable EER is an inner bound for the EER in a Gaussian broadcast channel (see Fig. 3(b)). We summarize this result in the following theorem.

Theorem 1: For a two-user Gaussian broadcast channel with power constraint P and noise power σ_1^2 and σ_2^2 for user 1 and user 2, respectively, an achievable EER is $EER(R_1, R_2) = EER_s(R_1, R_2) \cup EER_{ts}(R_1, R_2)$, where $EER_s(R_1, R_2)$ and $EER_{ts}(R_1, R_2)$ are given by

$$\begin{aligned} &EER_s(R_1, R_2) = \{(E_1, E_2) : \\ &E_1 \le \min\{E(R_1, \frac{\alpha P}{\sigma_1^2}), E_{t3}(R_1 + R_2, \frac{\alpha P}{\sigma_1^2}, \frac{(1-\alpha)P}{\sigma_1^2})\}, \\ &E_2 \le \min\{E(R_2, \frac{(1-\alpha)P}{\sigma_2^2}), E_{t3}(R_1 + R_2, \frac{\alpha P}{\sigma_2^2}, \frac{(1-\alpha)P}{\sigma_2^2})\}, \\ &0 < \alpha < 1\} \end{aligned} \quad (9)$$

$$\begin{aligned} &EER_{ts}(R_1, R_2) = \{(E_1, E_2) : \\ &E_1 \le \alpha E(\frac{R_1}{\alpha}, \frac{P}{\sigma_1^2}), E_2 \le (1-\alpha) E(\frac{R_2}{1-\alpha}, \frac{P}{\sigma_2^2}), 0 < \alpha < 1\}. \end{aligned} \quad (10)$$

■



(a) (b)

Fig. 3. Error exponent achievable region (a) time-sharing and superposition, (b) inner bound ($R_1 = R_2 = 0.5$; $\frac{P}{\sigma_1^2} = \frac{P}{\sigma_2^2} = 10$).

The proof of Theorem 1, as given in the Appendix, requires to show the existence of two codebooks, CB_1^* and CB_2^*, that *simultaneously* satisfy $P_{e1}^* \le e^{-NE_1}$ and $P_{e2}^* \le e^{-NE_2}$, for any pair of (E_1, E_2) in the achievable EER.

III. Improved Achievable Error Exponent Region by Naive Single-User Decoder

The result in Theorem 1 and in Fig. 3 is a little surprising. Since T. M. Cover's famous paper "Broadcast Channels" was published in 1972, it is believed that the superposition encoding is a better scheme than the time-sharing encoding [5]. When we consider unequal error protection for different users, however, the result in Fig. 3 suggests that the time-sharing scheme sometimes might be better than superposition. Although these results do not contradict those in [5] (since we are examining error exponents, while the work in [5] refers to channel capacity), there are three possible explanations to this observation:

(i) It might be the case that time-sharing can indeed expand the EER provided by superposition, especially for the case when one user requires a much better reliability than the other.

(ii) The achievable error exponents derived in Theorem 1 for superposition encoding use *joint* maximum likelihood (ML) decoders, i.e., decoding user 1's message based on the (i, j) pair maximizing $P(Y_1^N | X_{1i}^N, X_{2j}^N)$ and decoding user 2's message based on the (i, j) maximizing $P(Y_2^N | X_{1i}^N, X_{2j}^N)$. This is in general different and worse than using *individual* ML decoders, which minimize the error probability for user 1 and user 2, i.e., decoding user 1's message based on the i maximizing $\sum_j P(Y_1^N | X_{1i}^N, X_{2j}^N) P(X_{2j}^N)$ and decoding user 2's message based on the j maximizing $\sum_i P(Y_2^N | X_{1i}^N, X_{2j}^N) P(X_{1i}^N)$.

(iii) The third reason comes from the fact that in (3), (4), E_1^s and E_2^s are both upper bounded by E_{t3}, which accounts for the error event when both user 1's and user 2's codewords are decoded as wrong codewords. Since both E_1^s and E_2^s are upper bounded by E_{t3}. This might result in loose bounds which are derived using the *joint* ML decoder.

To answer this question it is desirable to find tight upper bounds for the optimal *individual* ML decoder. However, it seems that it is difficult to derive an analytical, single-letter expression for the error exponents using the individual ML decoders. Instead, we propose another decoding scheme, the *naive* single-user decoder, which can improve the error exponent region achieved by the joint ML decoders. In the naive single-user decoding, user 1 simply treats user 2 as noise, and user 2 also simply treats user 1 as noise. Since both users can choose either the joint ML decoders or the naive single-user decoders, the new error exponents for user 1 and user 2 using superposition encoding are

$$\begin{aligned} E_1^{'s} = \max\Bigg\{ &E(R_1, \frac{\alpha P}{(1-\alpha)P + \sigma_1^2}), \\ &\min\left[E(R_1, \frac{\alpha P}{\sigma_1^2}), E_{t3}(R_1 + R_2, \frac{\alpha P}{\sigma_1^2}, \frac{(1-\alpha)P}{\sigma_1^2})\right]\Bigg\} \end{aligned} \quad (11)$$

$$\begin{aligned} E_2^{'s} = \max\Bigg\{ &E(R_2, \frac{(1-\alpha)P}{\alpha P + \sigma_2^2}), \\ &\min\left[E(R_2, \frac{(1-\alpha)P}{\sigma_2^2}), E_{t3}(R_1 + R_2, \frac{\alpha P}{\sigma_2^2}, \frac{(1-\alpha)P}{\sigma_2^2})\right]\Bigg\}. \end{aligned} \quad (12)$$

Although decoding by treating the other user's interference as noise is sub-optimum, this simple scheme does improve the original EER achieved by E_1^s and E_2^s in (3), (4). In Fig. 4(a), the solid curve is the boundary of the original achievable EER by superposition using joint ML decoding, and the dashed curve (which merges with the solid curve at $(E_1, E_2) = (0.038, 0.002)$) is the boundary of the new achievable EER by superposition using the joint ML decoding and the naive single-user decoding. In Fig. 4(b), the solid curve is the boundary of the new achievable EER by superposition, and the dotted line is the achievable EER by time-sharing. For this operating point $(R_1, R_2) = (0.2, 0.65)$, the achievable EER by time-sharing is inside the achievable EER by superposition. In general, the EER defined by (11), (12) does not always expand the EER beyond that achieved by time sharing (this is the case in the example of Fig. 3).

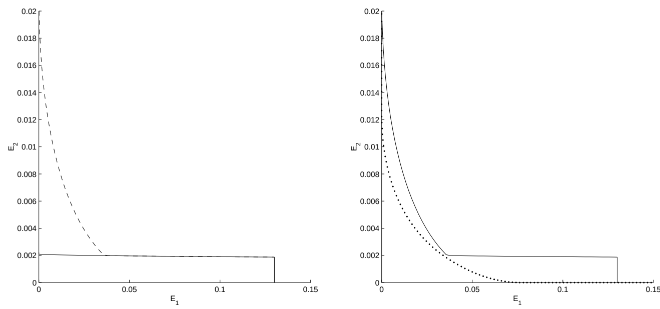


Fig. 4. Error exponent achievable region (a) superposition, (b) inner bound ($R_1 = 0.2, R_2 = 0.65$; $\frac{P}{\sigma_1^2}$ = 10, $\frac{P}{\sigma_2^2}$ = 5).

IV. Outer Bound for Error Exponent Region

In this section, we first derive an outer bound for the error exponent region of a discrete memoryless broadcast channel (DMBC), then we extend this result to a Gaussian broadcast channel.

Consider a DMBC defined by the joint probability mass function $P(Y_1, Y_2|X)$. The probability of decoding error for user i can always be lower bounded by the probability of decoding error for user i operating over a point-to-point channel defined by the marginal distribution $P(Y_i|X)$, for $i = 1, 2$. Further, we use the fact that the performance of a broadcast channel depends only on the marginal distributions $P(Y_1|X)$ and $P(Y_2|X)$, not on the joint distribution $P(Y_1, Y_2|X)$. To be specific, consider another DMBC with marginal distributions the same as those in the original DMBC, i.e., $P'(Y_1|X) = P(Y_1|X)$ and $P'(Y_2|X) = P(Y_2|X)$, but with $P'(Y_1, Y_2|X) \neq P(Y_1, Y_2|X)$ in general. The EER of this new DMBC is the same as the EER of the original DMBC, since the probability of error of each user depends only on the corresponding marginal distribution [6]. If we now allow the two receivers in the new DMBC to cooperate, we have a two-output single-user DMC, whose probability of error (using an optimal receiver), P'_e, should be less than or equal to the probability of system error P_e in the original DMBC. Using the union bound, it is also easy to show that $P_e \leq 2\max\{P_{e1}, P_{e2}\}$, where P_{ei} denotes the probability of error for user i in the original DMBC. Collecting all these ideas, we have the following outer bound for the EER.

$$E_1 \leq E_1^{su}(R_1) \tag{13}$$
$$E_2 \leq E_2^{su}(R_2) \tag{14}$$
$$\min\{E_1, E_2\} \leq \min_{P'(Y_1,Y_2|X)} E_{12}^{su}(R_1 + R_2), \tag{15}$$

where $E_i^{su}(R)$ denotes any valid error-exponent upper bound for a single-user channel defined by $P(Y_i|X)$, and $E_{12}^{su}(R)$ denotes any valid error-exponent upper bound for a single-input-two-output single-user channel defined by $P'(Y_1, Y_2|X)$, and the minimum on the right hand side of the last inequality is over all the distributions $P'(Y_1, Y_2|X)$ with the same marginal distributions as those of the original DMBC. If the original DMBC is a degraded broadcast channel (with user 1 having the better channel), then $\min_{P'(Y_1,Y_2|X)} E_{12}^{su}(R_1 + R_2) = E_1^{su}(R_1 + R_2)$.

The above argument can be easily extended to a Gaussian broadcast channel with power constraint P by noticing that a Gaussian broadcast channel is a degraded broadcast channel. We summarize the result in the following theorem.

Theorem 2: For a two-user Gaussian broadcast channel with power constraint P and noise power σ_1^2 and σ_2^2 for user 1 and user 2, an outer bound for the error exponent region $EER(R_1, R_2)$ is

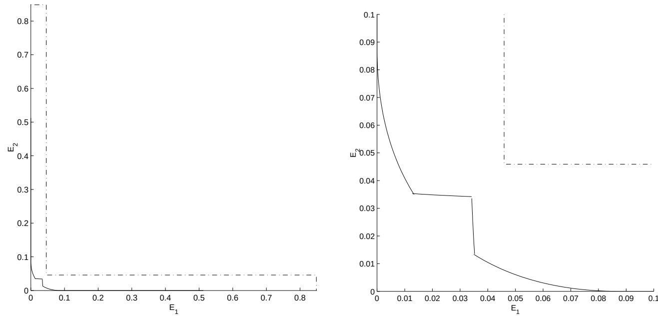
$$E_1 \leq E^{su}(R_1, \frac{P}{\sigma_1^2}) \tag{16}$$
$$E_2 \leq E^{su}(R_2, \frac{P}{\sigma_2^2}) \tag{17}$$
$$\min\{E_1, E_2\} \leq \max\{E^{su}(R_1 + R_2, \frac{P}{\sigma_1^2}), E^{su}(R_1 + R_2, \frac{P}{\sigma_2^2})\}, \tag{18}$$

where $E^{su}(R, SNR)$ is any upper bound for a single-user scalar Gaussian channel operating at rate R and having signal-to-noise ration SNR. ■

For illustration, we use the spherical packing bound for $E^{su}(R, SNR)$, though this bound can be improved by the minimum distance bound or the straight line bound at low rates [7], [8]. In Fig. 5(a), the small solid curve is the achievable error exponent region (the same curve in Fig. 3(b)), and the dash-dotted curve is the outer bound of the error exponent region. Fig. 5(b) is the same diagram as Fig. 5(a), but focuses on the region containing the inner bound.

V. Conclusion

In this paper, we consider an inner and an outer bound for the error exponent region in a Gaussian broadcast channel. Two simple strategies (time-sharing and superposition) are proposed to obtain achievable EERs. The concept of the EER is general and can be extended to other channel models, such as multiple access channels [9]. Currently the authors are



(a) (b)

Fig. 5. Inner and outer bounds for error exponent region ($R_1 = R_2 = 0.5$; $\frac{P}{\sigma_1^2} = \frac{P}{\sigma_2^2} = 10$).

investigating tighter inner and outer bounds for the EER and practical schemes to achieve these bounds.

VI. Appendix

We now prove that for any (E_1, E_2) interior to the achievable region $EER_s(R_1, R_2) \cup EER_{ts}(R_1, R_2)$, there exist codebooks CB_1^* and CB_2^* for user 1 and user 2 such that for any $\epsilon > 0$,

$$P_{e1}^* \leq e^{-N(E_1-\epsilon)} \tag{19}$$
$$P_{e2}^* \leq e^{-N(E_2-\epsilon)} \tag{20}$$

for sufficient large N, where N is the codeword length. For (E_1, E_2) interior to $EER_{ts}(R_1, R_2)$ or interior to $EER_s(R_1, R_2)$ using naive single-user decoding, the proof of the existence of the codebooks CB_1^* and CB_2^* is trivial. For (E_1, E_2) interior to $EER_s(R_1, R_2)$ using joint ML decoding, the proof is equivalent to showing that for any $\epsilon > 0$ and $0 < \alpha < 1$, there exist codebooks CB_1^* and CB_2^* for user 1 and user 2 satisfying the following inequalities

$$P_{e1}^* \leq \exp\left[-N\left(\min\left\{E(R_1, \frac{\alpha P}{\sigma_1^2}), E_{t3}(R_1+R_2, \frac{\alpha P}{\sigma_1^2}, \frac{(1-\alpha)P}{\sigma_1^2})\right\} - \epsilon\right)\right] \tag{21}$$
$$P_{e2}^* \leq \exp\left[-N\left(\min\left\{E(R_2, \frac{(1-\alpha)P}{\sigma_2^2}), E_{t3}(R_1+R_2, \frac{\alpha P}{\sigma_2^2}, \frac{(1-\alpha)P}{\sigma_2^2})\right\} - \epsilon\right)\right], \tag{22}$$

where P_{e1}^* and P_{e2}^* are the average probabilities of error for user 1 and user 2. Recall that $E(R, SNR)$ is the maximum of the random coding exponent $E_r(R, SNR)$ and the expurgated exponent $E_{ex}(R, SNR)$. Therefore, (21), (22) in fact implies four cases, each depending on whether the random coding exponent or the expurgated exponent dominates for user 1 or user 2. We prove in the appendix only the case when the random coding exponents dominate the expurgated exponents for both user 1 and user 2, i.e.,

$$P_{e1}^* \leq \exp\left[-N\left(\min\left\{E_r(R_1, \frac{\alpha P}{\sigma_1^2}), E_{t3}(R_1+R_2, \frac{\alpha P}{\sigma_1^2}, \frac{(1-\alpha)P}{\sigma_1^2})\right\} - \epsilon\right)\right] \tag{23}$$
$$P_{e2}^* \leq \exp\left[-N\left(\min\left\{E_r(R_2, \frac{(1-\alpha)P}{\sigma_2^2}), E_{t3}(R_1+R_2, \frac{\alpha P}{\sigma_2^2}, \frac{(1-\alpha)P}{\sigma_2^2})\right\} - \epsilon\right)\right]. \tag{24}$$

The proofs for the other three cases are similar.

In order to prove the existence of the codebooks CB_1^* and CB_2^* satisfying (23), (24), we construct two independent random codebooks CB_1 and CB_2, each with $M_1 = e^{NR_1}$ codewords and $M_2 = e^{NR_2}$ codewords respectively. Every element in CB_1 is independent and identically distributed with average power $\alpha P - \delta/2$, where δ is some positive number. Similarly, every element in CB_2 is independent and identically distributed with average power $(1-\alpha)P - \delta/2$. Suppose that user 1 sends codeword $c_{1,i}$ ($1 \leq i \leq M_1$) and user 2 sends codeword $c_{2,j}$ ($1 \leq j \leq M_2$). We use $c_{1,i'}$ to denote another codeword in CB_1 different from $c_{1,i}$, and use $c_{2,j'}$ to denote another codeword in CB_2 different from $c_{2,j}$. For any realization of the random codebooks CB_1 and CB_2, we define the following probabilities.

- P_{e11} : average of probability of error when user 1 decodes ($c_{1,i}$,$c_{2,j}$) as ($c_{1,i'}$,$c_{2,j}$)
- P_{e22} : average of probability of error when user 2 decodes ($c_{1,i}$,$c_{2,j}$) as ($c_{1,i}$,$c_{2,j'}$)
- P_{e13} : average of probability of error when user 1 decodes ($c_{1,i}$,$c_{2,j}$) as ($c_{1,i'}$,$c_{2,j'}$)
- P_{e23} : average of probability of error when user 2 decodes ($c_{1,i}$,$c_{2,j}$) as ($c_{1,i'}$,$c_{2,j'}$)
- P_{e1} : average probability of error for user 1
- P_{e2} : average probability of error for user 2

In general, all the above parameters are random variables. In the following, we first use Markov inequality to upper bound the tail probabilities for P_{e11}, P_{e22}, P_{e13}, and P_{e23}. Then we upper bound the probability of the event when the average transmitted power P_t is larger than the power constraint P. Finally, we use the union bound to prove the existence of the codebooks CB_1^* and CB_2^*.

A. Tail Probabilities for P_{e11}, P_{e12}, P_{e13}, and P_{e23}

For any random variable X, we use the notation $\overline{X}$ to denote the ensemble average $E\{X\}$. From Markov inequality, for any $\beta > 0$, we have

$$Pr\{P_{e11} > \beta\overline{P_{e11}}\} \leq \frac{1}{\beta} \tag{25}$$

$$Pr\{P_{e22} > \beta\overline{P_{e22}}\} \leq \frac{1}{\beta} \tag{26}$$

$$Pr\{P_{e13} > \beta\overline{P_{e13}}\} \leq \frac{1}{\beta} \tag{27}$$

$$Pr\{P_{e23} > \beta\overline{P_{e23}}\} \leq \frac{1}{\beta}. \tag{28}$$

In addition, we have the following inequalities using the random coding exponent argument

$$\overline{P_{e11}} \leq \exp\left[-NE_r\left(R_1, \frac{\alpha P - \frac{\delta}{2}}{\sigma_1^2}\right)\right] \tag{29}$$

$$\overline{P_{e22}} \leq \exp\left[-NE_r\left(R_2, \frac{(1-\alpha)P - \frac{\delta}{2}}{\sigma_2^2}\right)\right] \tag{30}$$

$$\overline{P_{e13}} \leq \exp\left[-NE_{t3}\left(R_1+R_2, \frac{\alpha P - \frac{\delta}{2}}{\sigma_1^2}, \frac{(1-\alpha)P - \frac{\delta}{2}}{\sigma_1^2}\right)\right] \tag{31}$$

$$\overline{P_{e23}} \leq \exp\left[-NE_{t3}\left(R_1+R_2, \frac{\alpha P - \frac{\delta}{2}}{\sigma_2^2}, \frac{(1-\alpha)P - \frac{\delta}{2}}{\sigma_2^2}\right)\right]. \tag{32}$$

B. Upper Bound for $Pr\{P_t > P\}$

For any realization of the random codebooks CB_1 and CB_2, denote $c_{1,i}(k)$ the kth element in the codeword $c_{1,i}$, and $c_{2,j}(k)$ the kth element in the codeword $c_{2,j}$. We also define the following notations

P_{ij} : average power of the codeword $(c_{1,i}, c_{2,j})$; $P_{ij} = \frac{1}{N}\sum_{k=1}^{N}[c_{1,i}(k) + c_{2,j}(k)]^2$

P_t : average transmitted power; $P_t = \frac{1}{M_1 M_2}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} P_{ij}$.

In general, P_{ij} and P_t are random variables. Since the random codebooks CB_1 and CB_2 are constructed independently with average power $\alpha P - \delta/2$ and $(1-\alpha)P - \delta/2$, we have $\overline{P_t} = P - \delta$. From the weak law of large numbers, $Pr\{|P_t - \overline{P_t}| > \delta\} < \delta$ for N sufficiently large. Therefore,

$$Pr\{P_t > P\} \leq Pr\{|P_t - \overline{P_t}| > \delta\} < \delta \tag{33}$$

for N sufficiently large.

C. Existence of the Codebooks CB_1^* and CB_2^*

Define $\mathbb{B}$ as the union of the following events

$$\mathbb{B} = \{P_{e11} > \beta\overline{P_{e11}}\} \cup \{P_{e22} > \beta\overline{P_{e22}}\} \cup \{P_{e13} > \beta\overline{P_{e13}}\} \cup \{P_{e23} > \beta\overline{P_{e23}}\} \cup \{P_t > P\}. \tag{34}$$

Therefore, we can use the union bound to get

$$Pr\{\mathbb{B}\} \leq 4\frac{1}{\beta} + \delta = \frac{4}{\beta} + \delta. \tag{35}$$

For arbitrary small δ, we can always choose β to get

$$Pr\{\mathbb{B}^c\} = 1 - Pr\{\mathbb{B}\} \geq 1 - \frac{4}{\beta} - \delta > 0. \tag{36}$$

Since $Pr\{\mathbb{B}^c\} > 0$, this implies that there exist codebooks CB_1^* and CB_2^* such that

- $P_{e11}^* \leq \beta\overline{P_{e11}}$, $P_{e22}^* \leq \beta\overline{P_{e22}}$, $P_{e13}^* \leq \beta\overline{P_{e13}}$, $P_{e23}^* \leq \beta\overline{P_{e23}}$.
- $P_t^* \leq P$,

where P_{e11}^*, P_{e22}^*, P_{e13}^*, P_{e23}^*, and P_t^* are the parameters corresponding to the codebooks CB_1^* and CB_2^*. Because $P_{e1}^* \leq P_{e11}^* + P_{e13}^*$, $P_{e2}^* \leq P_{e22}^* + P_{e23}^*$, and $\overline{P_{e11}}$, $\overline{P_{e22}}$, $\overline{P_{e13}}$, $\overline{P_{e23}}$ are upper bounded by (29), (30), (31), (32), it is easy to see that

$$P_{e1}^* \leq \exp\left[-N\left(\min\left\{E_r(R_1, \frac{\alpha P}{\sigma_1^2}), E_{t3}(R_1+R_2, \frac{\alpha P}{\sigma_1^2}, \frac{(1-\alpha)P}{\sigma_1^2})\right\} - \epsilon\right)\right] \tag{37}$$

$$P_{e2}^* \leq \exp\left[-N\left(\min\left\{E_r(R_2, \frac{(1-\alpha)P}{\sigma_2^2}), E_{t3}(R_1+R_2, \frac{\alpha P}{\sigma_2^2}, \frac{(1-\alpha)P}{\sigma_2^2})\right\} - \epsilon\right)\right], \tag{38}$$

for sufficiently large N since E_r and E_{t3} are continuous functions. This completes the proof.

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