A SHORT PROOF OF THE FACTOR THEOREM FOR FINITE GRAPHS

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We define a *graph* as a set V of objects called *vertices* together with a set E of objects called *edges*, the two sets having no common element. With each edge there are associated just two vertices, called its *ends*. We say that an edge *joins* its ends. Two vertices may be joined by more than one edge.

A *subgraph* G' of a graph G is a graph whose edges and vertices are edges and vertices respectively of G and in which each edge has the same ends as in G. If S is any set of vertices of G we denote by G_S the subgraph of G whose vertices are the vertices of G not in S and whose edges are the edges of G not having an element of S as an end.

A graph is *finite* if V and E are both finite and *infinite* otherwise. In this paper we consider only finite graphs.

Suppose given a finite graph G. For $a \in V$ and $A \in E$ we write $e(A, a) = 1$ if a is an end of A and $e(A, a) = 0$ otherwise. Let f be a function which associates with each vertex a of G a unique positive integer $f(a)$. We say that G is *f-soluble* for a given f if to each $A \in E$ we can assign a non-negative integer $g(A)$ such that

$$\sum_A e(A, a)\, g(A) = f(a) \tag{1}$$

for each $a \in V$. If E is null but V is not null, we consider that G is not f-soluble for any f. We ignore the case in which V and E are both null (when G is the *null graph*).

It may be possible to solve (1) so that $g(A) = 0$ or 1 for each A. Then we call the subgraph of G whose vertices are the vertices of G and whose edges are those edges A of G for which $g(A) = 1$ an *f-factor* of G. Thus an f-factor of G is a subgraph of G such that each $a \in V$ is a vertex of the subgraph and an end of just $f(a)$ edges of the subgraph.

If n is any positive integer we define an *n-factor* of G as an f-factor such that $f(a) = n$ for each a.

Necessary and sufficient conditions are known for f-solubility, for the existence of an f-factor and for the existence of a 1-factor. We state these as Theorems A, B, and C after a few preliminary definitions.

The *degree* $d(a)$ of a vertex a of G is the number of edges of G having a as an end. If $S \subseteq V$ and $a \in V - S$ we denote the degree of a in G_S by $d_S(a)$.

Suppose $S \subseteq V$. We write $\alpha(S)$ for the number of vertices of S. The graph G_S is uniquely decomposable into disjoint connected parts which we call *components*. (Hassler Whitney **(6)** uses the term *connected pieces*, and König

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(2) *zusammenhängende Bestandteile.*) We write $h_u(S)$ for the number of components of G_S for which the number of vertices is odd. We write $T(S)$ for the set of vertices of $V - S$ which are joined only to vertices of S.

We denote by $q(S)$ the number of components C of G_S for which there is more than one vertex and

$$\sum_{a \in C} f(a) \equiv 1 \pmod 2. \tag{2}$$

Here we write $a \in C$ to denote that a is a vertex of C.

Now suppose $T \subseteq V - S$. If C is a component of $G_{S \cup T}$ we denote by $v(C)$ the number of edges of G having one end a vertex of C and the other an element of T. We denote by $q(S, T)$ the number of components C of $G_{S \cup T}$ such that

$$v(C) + \sum_{a \in C} f(a) \equiv 1 \pmod 2. \tag{3}$$

THEOREM A. *G is without a 1-factor if and only if there is a subset S of V such that*

$$h_u(S) > \alpha(S). \tag{4}$$

THEOREM B. *G is not f-soluble if and only if there is a subset S of V such that*

$$\sum_{a \in S} f(a) < q(S) + \sum_{c \in T(S)} f(c). \tag{5}$$

THEOREM C. *G is without an f-factor if and only if there is a subset S of V and a subset T of $V - S$ such that*

$$\sum_{a \in S} f(a) < q(S, T) + \sum_{c \in T} (f(c) - d_S(c)). \tag{6}$$

A short proof of Theorem A has been given by the author **(4)**. Maunsell **(3)** has improved it by substituting a piece of elementary graph theory for an appeal to the theory of determinants. Theorem B is readily deducible from Theorem C; details are given in **(5)**. However, proofs of Theorem C, even in the special case dealing with n-factors, have hitherto been long and complicated **(1; 5)**. In this paper we present a comparatively short argument whereby Theorem C is deduced as a consequence of Theorem A.

Deduction of Theorem C from Theorem A. Suppose first that G has a vertex a such that $d(a) < f(a)$. Then G can have no f-factor. Moreover (6) is satisfied with $S = 0$ and $T = \{a\}$. Thus Theorem C is trivially true in this case.

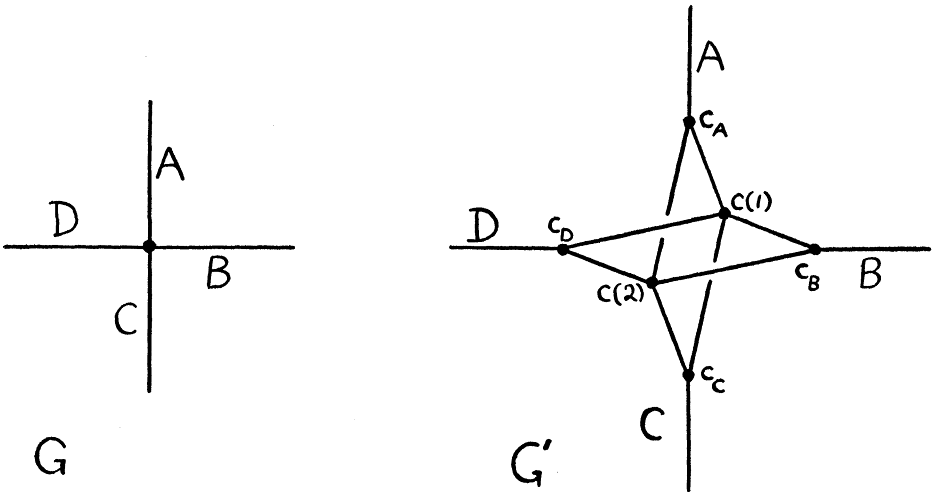
In the remaining case we have $d(a) \geqslant f(a)$ for each $a \in V$. We write $s(a) = d(a) - f(a)$.

Given any sufficiently large set Q we define a graph G' whose vertices are elements of Q in the following way. With each $c \in V$ we associate $d(c)$ distinct elements c_A of Q, one for each edge A of G such that $e(A, c) = 1$, and $s(c)$ other distinct elements $c(1), c(2), \ldots, c(s(c))$ of Q. We denote the sets of the $d(c)$ elements c_A and the $s(c)$ elements $c(i)$ by $X(c)$ and $Y(c)$ respectively. We postulate that the two sets $X(c) \cup Y(c)$ defined for two distinct elements c of V shall have no common element. The set V' of vertices of G' is given by

$$V' = \bigcup_{c \in V} (X(c) \cup Y(c)). \tag{7}$$

For any edge A of G, with ends x and y say, we postulate that G' has just one edge joining x_A and y_A. We denote this also by the symbol A. We further postulate that for each $c \in V$ each element of $X(c)$ is joined to each member of $Y(c)$ by just one edge of G', and that G' has no edges other than those required by these two rules.

For each $c \in V$, the elements of $X(c) \cup Y(c)$ and the edges of G' joining them constitute a subgraph, $\mathrm{St}(c)$, of G', which we call the *star-graph of c in G'*. $\mathrm{St}(c)$ is connected if $s(c) > 0$, and in the case $s(c) = 0$ only if $d(c) = f(c) = 1$. The diagram shows a star-graph $\mathrm{St}(c)$ for the case $d(c) = 4$ and $f(c) = 2$. (The edges A B C and D in this diagram do not belong to $\mathrm{St}(c)$.)



LEMMA. *G has an f-factor if and only if G' has a* 1*-factor.*

Proof. If G has an f-factor let F be its set of edges and let F' be the set of edges of G' denoted by the same letters. For each $c \in V$ we adjoin to F' exactly $s(c)$ edges joining the $s(c)$ vertices of $Y(c)$ to the $s(c)$ vertices of $X(c)$ which are not ends of edges of F'. By the definition of G' we can do this without introducing into F' two edges with a common end. We thus construct a 1-factor of G'.

Conversely suppose G' has a 1-factor whose set of edges is H. Let H_0 be the set of edges of H whose two ends are vertices of distinct star-graphs $\mathrm{St}(c)$. For each $c \in V$ just $s(c)$ elements of H have an end in $Y(c)$ and therefore just $d(c) - s(c) = f(c)$ elements of H_0 have an end in $X(c)$. It follows that the edges of G corresponding to the members of H_0 define an f-factor of G.

A subset W of V' will be called *simple* if it satisfies the following conditions for each $a \in V$:

(i) If $X(a) \cap W \neq 0$ then $X(a) \subseteq W$,

(ii) If $Y(a) \cap W \neq 0$ then $Y(a) \subseteq W$,

(iii) At most one of $X(a)$ and $Y(a)$ is a subset of W.

Condition (iii) implies that $X(a)$ cannot be a subset of W when $Y(a)$ is the null set, i.e., when $d(a) = f(a)$.

Consider any simple subset W of V'. We write S and T for the sets of vertices c of G such that $X(c) \subseteq W$ and $Y(c) \subseteq W$ respectively. The sets S and T are disjoint. We have

(8) $$\alpha(W) = \sum_{c \in T} (d(c) - f(c)) + \sum_{a \in S} d(a).$$

Let H be any component of G'_W.

It may happen that H has just one vertex, which is of the form c_A. Then $c \in T$ and the end of A in G other than c belongs to S. The number of such components H is the number of edges A of G having one end in S and the other in T, that is

$$\sum_{c \in T} (d(c) - d_S(c)).$$

Another possibility is that H has just one vertex, which is of the form $c(i)$. The number of such components is

$$\sum_{a \in S} (d(a) - f(a)).$$

In the remaining case, H has at least one edge. If H has no edge in common with one of the star-graphs $\mathrm{St}(a)$ it must consist of a single edge with its two ends. Then the number of vertices of H is even. If H has an edge in common with $\mathrm{St}(a)$ then $Y(a) \neq 0$ and so $\mathrm{St}(a)$ is connected. Moreover $\mathrm{St}(a)$ is then a subgraph of H. A component of G'_W having a connected star-graph $\mathrm{St}(a)$ with at least one edge as a subgraph will be called *large*.

Suppose H is large. Let M be the set of all vertices a of G such that $\mathrm{St}(a)$ is a connected subgraph of H with at least one edge. Then $M \subseteq V - (S \cup T)$. H is made up of these star-graphs $\mathrm{St}(a)$, a set N of edges which link them to form a connected graph H_0 and a set P of edges having one end a vertex of H_0 and one end a vertex c_A such that $c \in T$. Clearly M is the set of vertices of a component $K(H)$ of $G_{S \cup T}$. We may think of $K(H)$ as derived from H_0 by shrinking each of the star-graphs $\mathrm{St}(a)$, $a \in M$, to a single vertex. Conversely suppose K is any component of $G_{S \cup T}$. If c is a vertex of K then $Y(c) \neq 0$ since $c \notin T$ and therefore $\mathrm{St}(c)$ is connected and has at least one edge. This star-graph is a subgraph of a large component H of G'_W and we must have $K = K(H)$.

For a large component H of G'_W having just n vertices

$$n = \sum_{a \in K(H)} \{d(a) + (d(a) - f(a))\} + v(K(H))$$
$$\equiv \sum_{a \in K(H)} f(a) + v(K(H)) \pmod 2.$$

Hence the number of large components of G'_W for which the number of vertices is odd is $q(S, T)$.

Using (8) we obtain the formulae

$$h_u(W) = q(S, T) + \sum_{a \in S} (d(a) - f(a)) + \sum_{c \in T} (d(c) - d_S(c)), \tag{9}$$

$$\begin{aligned} h_u(W) &- \alpha(W) \\ &= q(S, T) - \sum_{a \in S} f(a) + \sum_{c \in T} (f(c) - d_S(c)). \end{aligned} \tag{10}$$

The quantities on the left in these equations are defined in terms of G', those on the right in terms of G.

Suppose there are disjoint subsets S and T of V satisfying (6). Select two such subsets so that $\alpha(S)$ has the least possible value. Assume that $f(b) = d(b)$ for some $b \in S$. If we replace S by $S - \{b\}$ and T by $T \cup \{b\}$ inequality (6) will remain valid, for with at most $d(b)$ exceptions the numbers $v(C)$ associated with the components of $G_{S \cup T}$ are unaltered. This contradicts the definition of S. Hence $f(b) < d(b)$ for each $b \in S$. Let W be the union of the sets $X(a)$ such that $a \in S$ and the sets $Y(c)$ such that $c \in T$. Then W is simple since $Y(c)$ is non-null when $X(c) \subseteq W$. It follows from (10) that $h_u(W) > \alpha(W)$ in G'. Hence G' has no 1-factor, by Theorem A. Hence G has no f-factor, by the Lemma.

Conversely suppose G has no f-factor. Then by Theorem A and the Lemma there is a set W of vertices of G' such that $h_u(W) > \alpha(W)$. Choose such a W so that $\alpha(W)$ has the least possible value.

Suppose there exists $a \in V$ such that $Y(a) \cap W \neq 0$ and $Y(a) \cap (V' - W) \neq 0$. Write $Z = W - (Y(a) \cap W)$. Then G'_W and G'_Z differ in one component only, provided that $X(a)$ is not a subset of W, since the members of $Y(a)$ are all joined to the same vertices of G'. If $X(a) \subseteq W$ then each component of G'_W is a component of G'_Z. In either case we have $h_u(Z) \geqslant h_u(W) - 1$ and $\alpha(Z) \leqslant \alpha(W) - 1$. Hence $h_u(Z) - \alpha(Z) \geqslant h_u(W) - \alpha(W)$, contrary to the definition of W. We deduce that $Y(a) \subseteq W$ if $Y(a) \cap W \neq 0$.

Suppose next that $X(a) \cap W \neq 0$. Choose $b \in X(a) \cap W$. Write $Z = W - \{b\}$. There is at most one component of G'_W which has a vertex not a member of $Y(a)$ joined to b in G'. Hence if $Y(a)$ is contained in W the numbers $h_u(Z)$ and $h_u(W)$ can differ by at most one. Then $h_u(Z) \geqslant h_u(W) - 1$, $\alpha(Z) = \alpha(W) - 1$ and therefore $h_u(Z) - \alpha(Z) \geqslant h_u(W) - \alpha(W)$. This contradicts the definition of W. We deduce that, for the case $X(a) \cap W \neq 0$, $Y(a)$ is not a subset of W and therefore $Y(a) \cap W = 0$ by the result of the preceding paragraph. This proves that $X(a)$ and $Y(a)$ cannot both be subsets of W, since $X(a)$ is never null. ($d(a) \geqslant f(a) > 0$.)

Suppose both $X(a) \cap W$ and $X(a) \cap (V' - W)$ are non-null. We choose $b \in X(a) \cap W$ and write $Z = W - \{b\}$ as before. Since $Y(a) \cap W = 0$ all the vertices of $Y(a)$ belong to one component of G'_W, for each is joined in G' to each vertex of $X(a) \cap (V' - W)$. But there is at most one component of

G'_W which has a vertex not a member of $Y(a)$ joined to b in G'. Hence with at most two exceptions the components of G'_W are components of G'_Z. Accordingly

$$h_u(Z) \geqslant h_u(W) - 2,$$
$$h_u(Z) - \alpha(Z) \geqslant h_u(W) - \alpha(W) - 1.$$

But $h_u(Z)$ is by definition the number of components of G'_Z having an odd number of vertices. Hence

$$h_u(Z) + \alpha(Z) \equiv \alpha(V') \pmod 2$$

and similarly

$$h_u(W) + \alpha(W) \equiv \alpha(V') \pmod 2.$$

We may write these results as

$$h_u(Z) - \alpha(Z) \equiv \alpha(V') \equiv h_u(W) - \alpha(W) \pmod 2.$$

Hence $h_u(Z) - \alpha(Z) \geqslant h_u(W) - \alpha(W)$ and so the definition of W is contradicted. We deduce that $X(a) \subseteq W$ if $X(a) \cap W \neq 0$.

We have now proved that W is simple. We define S and T in terms of W as before. Using (10) we find that S and T satisfy (6).

This completes the proof of Theorem C.

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