THE ASSIGNMENT PROBLEM[1]

BY

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1. Introduction. Let $a = (a_{ij})$ be an n by n matrix of real numbers. To find *a permutation* (p_i) *such that* Σa_{ip_i} *be as great as possible* is the assignment problem. It seems to have first appeared in mathematical theory [1]; then, independently, as a computational problem [2]. For the subsequent, again independent, development and for the name of the problem see [3].

To compute $n!$ sums of n terms and to choose a greatest sum is mathematically trivial; there remains a typical problem of numerical analysis: to find the solution, or an approximation of it, in a fast and convenient way, even for large n, or for a large set of matrices.

The material in this paper will be presented under the following subheadings:

Normalization
Approximate solution
Vertex-approach and face-approach methods
The equidistribution problem
Related problems

The material presented under the next-to-last heading is also of purely mathematical interest; it and parts of other sections are new. A summary of geometric concepts used is appended.

It is convenient to look at a as a point or vector in real n^2-space. Defining the permutation point $p = (p_{ij})$ by $p_{ij} = 1$ for $j = p_i$ and $p_{ij} = 0$ for $j \neq p_i$, our problem is to maximize $\Sigma_{ij} a_{ij} p_{ij}$ or equivalently $\cos(a,p)$. The linear subspace L_1 spanned by the $n!$ points p is clearly the $(n-1)^2$-space defined by

$$\Sigma_i a_{ij} = \Sigma_j a_{ij} = 1;$$

its parallel L_0 through 0 is defined by $\Sigma_i a_{ij} = \Sigma_j a_{ij} = 0$. The orthogonal complement is the $(2n-1)$-space L_0' consisting of all $a = (a_{ij})$, $a_{ij} = \lambda_i + \mu_j$, and meets L_1 at the point c, $c_{ij} = 1/n$, centroid of the points p. The *permutation polyhedron* or convex hull P of the points p is [4] given in L_1 by its n^2 faces $a_{ij} \geqq 0$ (just for $n = 1$ and 2 the number of faces is 0 and 2, see Sec. 4); its points are the doubly stochastic matrices.

2. Normalization. Among transformations that do not change the solving permutation we mention:

(1) Replacing a_{ij} by $e^{a_{ij}}$ and Σ by Π leads to the problem of finding the

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absolutely greatest term of a determinant. The problem of finding the, without qualification, greatest term (for $n \geqq 3$ necessarily $\geqq 0$) is slightly different.

(2) Replacing a by $-a$ leads to the problem of finding a *smallest* permutation sum.

(3) We may replace a by λa, $\lambda > 0$. Hence all elements can be supposed to be absolutely $\leqq 1$.

(4) An arbitrary constant may be added to all elements of the same row or column.

By repetition of the last device (that is, adding an element of L_0') we can achieve one of the following preconditionings.

(4*a*) Make all elements nonnegative, or positive.

(4*b*) Make all elements nonnegative, and have at least one zero in every row; thereafter also in every column. (Alternating between rows and columns will take, on the average, more steps.) If we now find a permutation formed by zeros (this, the *ideal assignment problem* [7], may itself not be easy), it will

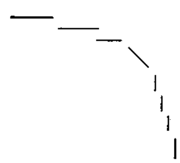

FIG. 1

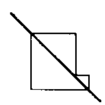

FIG. 2

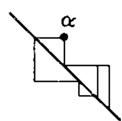

FIG. 3

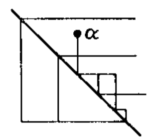

FIG. 4

In Figs. 2 to 4 for the sake of clarity the permutation sum under consideration is the main diagonal.

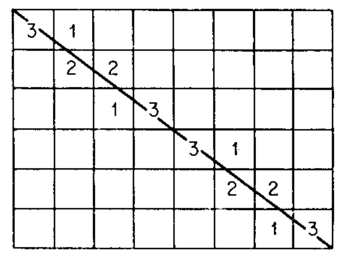

FIG. 5

give a least permutation sum. In any event, the rows and columns can now (again, not always easily) be rearranged so that a subset of zeros lies on a monotone and convex curve connecting two corners, that is, it consists of $a_{11}, \cdots, a_{1,m_1}, a_{2,m_1+1}, \cdots, a_{2,m_2}, \cdots, a_{k,m_k}, a_{n-n_l,n+1-l}, \cdots, a_{n-n_{l-1}+l,n+1-l}, \cdots, a_{n-n_1,n}, \cdots, a_{n,n}$, where $m_1 \geqq m_2 - m_1 \geqq \cdots \geqq m_k - m_{k-1}$, $n_1 \geqq n_2 - n_1 \geqq \cdots \geqq n_l - n_{l-1}$, $k + 1 = n - n_l$, $m_k = n - l$ (Fig. 1). (The number of such curves in a square or rectangle is related to the partition functions of additive number theory.)

(4*c*) Make all elements of the first row and column zero. This normalization is unique but unsymmetric.

(4*d*) Make all row sums and column sums zero. This normalization (amounting to the decomposition a + element of L_0' = element of L_0) is unique, symmetric, and still very simple. Alternatively, choose

$$a + \text{element of } L_0' + c = \text{element of } L_1.$$

(4*e*) After step 4*d*, add a constant to all elements so as to make them nonnegative but at least one of them zero. Thereafter multiply by a constant either so as to make the maximum element and thus also the spread 1, or, preferably, so as to make the common value of all row sums and column sums 1, thereby obtaining an equivalent element of P with maximum spread.

(4f) Make the spread $\max a_{ij} - \min a_{ij}$ a minimum. It is easily seen that the horizontal spread $\max_i (\max a_{ij} - \min a_{ij})$ and the vertical spread are then also minimal. Or minimize the horizontal spread sum $\Sigma_i(\max a_{ij} - \min a_{ij})$ or the vertical spread sum. These are special linear minimization problems, the first of which allows a simple minimum criterion involving circuits.

3. Approximate solution (cf. [3]). For shortening some of the direct methods below it may be desired to find a lower bound for the greatest permutation sum. Any permutation sum is such a bound and may solve a modified problem like approximating, in some sense, the greatest sum or surpassing a certain threshold value.

$\Sigma_i \min a_{ij}$ is an easily computable lower bound for all permutation sums. The quality of any lower bound can be appraised by comparing with the upper bound $\Sigma_i \max a_{ij}$.

The fact that every permutation sum is itself a lower bound leads to the proposal [9] to choose permutations at random. To get more permutations per second examined, one may change them gradually by interchanging each time two indices. It is of course advisable to use an inbuilt random-number-generation procedure [10]. As for further variants, one can allow exchanges of three or more indices and/or restrict changes to those for which the sum is raised or not too much dropped.

Among any n complementary permutations $p^{(k)}$ (that is, $\Sigma_k p_{ij}^{(k)} = 1$) at least one sum is $\geqq \Sigma_{ij} a_{ij}/n$ (for doubly stochastic matrices, 1).

The greatest and smallest permutation sum may, of course, have common elements.

(a) A good lower bound, and one that is often naïvely surmised to be the solution, is $s = \Sigma a_{ip_i}$ where $a_{ip_i} = \max_{j \neq p_1, \ldots, p_{i-1}} a_{ij}$. For doubly stochastic matrices $s = \Sigma a_{ij_i} \geqq \Sigma_{ip_n} = 1$.

The bound s depends on the arrangement of rows and, for a certain arrangement (and choice of $\max a_{ij}$ in case of ambiguity), does yield the solution. For let Σa_{ip_i} be a greatest permutation sum. Have row i precede row j if $a_{jp_i} > a_{jp_j}$. This is feasible, since a contradiction like $a_{jp_i} > a_{jp_j}$, $a_{kp_j} > a_{kp_k}$, $a_{ip_k} > a_{ip_i}$ (a "circuit," Fig. 2) would show how to replace Σa_{ip_i} by a greater sum. Though the partial order indicated may uniquely determine the order of the rows, as in

$$\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}, \quad \text{or} \quad \frac{1}{72}\begin{pmatrix} 45 & 7 & 20 \\ 27 & 26 & 19 \\ 0 & 39 & 33 \end{pmatrix} \varepsilon P, \quad \text{or} \quad \frac{1}{34}\begin{pmatrix} 12 & 9 & 9 & 4 \\ 13 & 12 & 0 & 9 \\ 0 & 13 & 12 & 9 \\ 9 & 0 & 13 & 12 \end{pmatrix} \varepsilon P,$$

it usually does not, which makes it seem worth while to try random arrangements of rows for s.

Since we see that every greatest permutation sum must contain a maximal element of some row, the sum can also be obtained by taking, after rearrangement of rows and columns, the greatest element of the first row, then, after deleting its column and row, the greatest element in the first column, alternating between rows and columns or using some other similar prescription.

A natural specialization of s, which is independent of the arrangement of rows and unaffected by transposing a, is

$$t = \Sigma a_{p_k q_k},$$

where

$$a_{p_k q_k} = \max_{i \neq p_1, \cdots, p_{k-1}; j \neq q_1, \cdots, q_{k-1}} a_{ij}.$$

For positive a it is easy to see that $t > \Sigma_{ij} a_{ij}/(2n + 1)$. Though t, even for $a \varepsilon P$, need not give the solution, as shown by the two last numerical examples, t seems to be a good bet or start for $a \varepsilon L_1$. For $n \leqq 3$ and $a \varepsilon L_1$, every greatest permutation sum must contain $\max a_{ij}$; indeed,

$$\tfrac{1}{3}((a_{11} + a_{22} + a_{33}) - (a_{12} + a_{23} + a_{31})) = a_{11} - a_{23} = a_{22} - a_{31} = a_{33} - a_{12}$$

is easy to verify and entails just the statement made.

The t-procedure requires more comparisons than the s-procedure. The results of both procedures are, for a general two by two matrix, on the average equally good; for a three by three matrix reduced as in Sec. 2(4b), the s-procedure gives, on the average, slightly better results.

An advantage of the t-procedure is that it also furnishes an upper bound. In fact, an easy induction shows that, for positive a, $2t$ surpasses every permutation sum [11]. This can be a better bound than $\Sigma_i \max a_{ij}$, for example, for $a \varepsilon P$ with $a_{i1} = 3/n$ or 0 as $i \leqq n/3$ or $i > n/3$, $a_{i3} = 3/n$ or 0 as $i > 2n/3$ or $i \leqq 2n/3$, $a_{i1} + a_{i2} + a_{i3} = 3/n$, $a_{ik} = 1/n$ for $k > 3$, $n/3$ an integer $\geqq 3$. The s-procedure has no corresponding property: choosing $a \varepsilon P$ with all a_{ii}, a_{1i}, a_{in}, and $a_{1n} = 1/n$, all $a_{i,i-1} = 1 - 2/n$, all other $a_{ik} = 0$, we obtain $s = 1$ (this choice can be enforced by slightly altering a) but a greatest sum $n - 3 + 3/n$.

To obtain an upper bound it is not necessary to find all of t. In fact, the sum of the first k of the $2n$ terms $a_{p_1 q_1}, a_{p_1 q_1}, a_{p_2 q_2}, a_{p_2 q_2}, \cdots, a_{p_n q_n}, a_{p_n q_n}$ is not less than every sum of k elements of a belonging to a permutation; this can again be verified by induction on n. For positive a the bound t' pertaining to $k = n$ is obviously $< 2t$, even $< 2t''$, where t'' is any permutation sum coinciding with t in its first half.

However, t' can be $< u = \Sigma_i \max a_{ij}$ only for $n \geqq 3$, and if $a \varepsilon P$, only for $n \geqq 5$, for example, for

$$\frac{1}{225}\begin{pmatrix} \underline{63} & 6 & 52 & 52 & 52 \\ 0 & 57 & \underline{56} & 56 & 56 \\ 54 & \underline{54} & 39 & 39 & 39 \\ 54 & 54 & 39 & \underline{39} & 39 \\ 54 & 54 & 39 & 39 & \underline{39} \end{pmatrix},$$

with $u = \frac{282}{225}$, $t' = \frac{279}{225}$, $t = \frac{237}{225}$, greatest sum (underlined) $= \frac{251}{225}$. For $n = 4$ let $t' = 2a_{11} + 2a_{22}$, $a_{11} \geqq a_{22}$. Clearly $t' \geqq u$ if $\max_j a_{ij} > a_{i1}$ for more than one i; but if $\max_j a_{ij} = a_{i1}$ for $i \neq i_0$, then $u = \max_j a_{i_0 j} + \Sigma_{j>1} a_{i_0 j}$ is again $\leqq t'$. Similarly for $n < 4$.

(*b*) The matrix a may itself be replaced by an approximation or derived matrix a'. Any a' for which $a'_{ij} > a'_{ij'}$ if and only if $a_{ij} > a_{ij'}$ leads to the same s-permutations as a; retention of the greater-than relation over the whole matrix preserves the t-permutations. However, these relations are of little relevance since addition of an element of L'_0 destroys them. Not so for $a_{ij} + a_{i'j'} > a_{ij'} + a_{i'j}$, the assignment problem for every two by two submatrix; in fact, knowledge of this relation enables one, for $n = 3$, to determine either the greatest permutation sum, or at least the two greatest ones. One may [12] give up even more information by defining $a'_{ij} = \Sigma_{i',j'} 1$ for the above pairs i', j' (scoring $\frac{1}{2}$ for equality); then $2n^{-1}(n-1)^{-2} a' \in P$. For $n \leqq 3$ no information is lost, and $a'' = (a')' = a'$. An example with $a'' \neq a'$ is

$$a = \frac{1}{41}\begin{pmatrix} 5 & 12 & 11 & 13 \\ 12 & 23 & 6 & 0 \\ 11 & 6 & 13 & 11 \\ 13 & 0 & 11 & 17 \end{pmatrix} \varepsilon P,$$

$$a' = \begin{pmatrix} 1 & 6 & 5 & 6 \\ 6 & 9 & 3 & 0 \\ 5 & 3 & 6 & 4 \\ 6 & 0 & 4 & 8 \end{pmatrix},$$

$$a'' = a''' = \begin{pmatrix} 0 & 7 & 5 & 6 \\ 7 & 8 & 3 & 0 \\ 5 & 3 & 6 & 4 \\ 6 & 0 & 4 & 8 \end{pmatrix}.$$

(*c*) The very crudest approximation $n \max a_{ik}$ may still have a great probability of being correct. Consider a matrix a of nonnegative elements, each of which, independently of the others, has a probability $1 - q > 0$ of being 0. What is the probability $1 - q'$ that the least sum is $n \min a_{ik} = 0$? It is known [7] that the least sum is 0 unless there exists a "block" of nonzero elements filling the intersections of some i rows and $n - i + 1$ columns, $1 \leqq i \leqq n$. The probability for a block at a given location is $q^{i(n-i+1)}$, and the probability of any i block's occurring is therefore at most

$$q_i = \binom{n}{i}\binom{n}{i-1} q^{i(n-i+1)}.$$

Hence $q' \leqq \Sigma q_i$. Now we shall show that for $nq^{n/2} < \frac{1}{2}$ (a fortiori for $n > -c_1(1-q)\ln(1-q) + c_2$ with universal constants c_1 and c_2), $q_i \leqq q_1 = nq^n$, so that $q' \leqq n^2 q^n < \frac{1}{4}$ and $q' \to 0$ for $n \to \infty$.

Since $q_i = q_{n+1-i}$, we can suppose $2i \leqq n + 1$. We show that

$$\frac{q_{i+1}}{q_i} = (n - i + 1)(n - i)(i + 1)^{-1}i^{-1}q^{n-2i} < 1$$

for $2i \leqq n - 1$. For $n = 2i + 1$, this amounts to

$$l_i = (1 + 2i^{-1})^{i/2} < (4i + 2)^{i/(2i+1)} = r_i,$$

true for $i = 1$ and 2; for $i > 2$, $l_i \uparrow e < r_3 \leqq r_i$. (The symbol $\uparrow$ means "increases and converges to.") Now use induction on n for fixed i to prove $q_{i+1}/q_i < 1$, rewritten as

$$l'_n = \left(1 - \frac{(i-1)}{n}\right)^n \left(1 - \frac{i}{n}\right)^n (2n)^{4i} < 4^n i^n (i + 1)^n = r'_n.$$

It suffices to prove $l'_{n+1}/l'_n < r'_{n+1}/r'_n$ where the three factors of the left side decrease with increasing n (convexity of the logarithmic function) whereas the right side remains $4i(i + 1)$. Thus we have only to verify

$$\frac{l'_{2i+2}}{l'_{2i+1}} < \frac{r'_{2i+2}}{r'_{2i+1}},$$

which can be written

$$l''_i = \left(1 + \frac{2}{i+1}\right)^{2i+2} < 16\,\frac{i}{i+\frac{1}{2}}\,\frac{(i+1)^2}{(i+\frac{1}{2})(i+2)}\,(i + 1)^2 = r''_i$$

and confirmed for $i = 1$; for $i > 1$, $l''_i \uparrow e^4 < r''_2 \leqq r''_i$.

This justification of the simplest estimate for the sum can be completed by a justification of the before-mentioned simplest procedures for finding a maximizing permutation, thus determining the circumstances under which the usual sweeping elimination of all more sophisticated methods is safe, a consideration of fundamental importance in practical application. Suppose $0 \leqq a_{ij} < m$, a fixed integer, and define a' by $a'_{ij} = [a_{ij}]$, the greatest integer $\leqq a_{ij}$. Assuming that a'_{ij}, independently of every other element, is $0, 1, \cdots, m - 1$ with equal probability $1/m$, what are the expected value e of s (defined in Sec. 3(a), with min for max) and the probability q that $s = 0$? Evidently $q = \Pi_i(1 - 1/m)^i$, with a positive limit for $n \to \infty$. And since $e(n) - e(n - 1)$ is easily seen to be

$$(m - 1)m^{-n} + (m - 2)(2^n - 1)m^{-n} + \cdots + 2((m - 2)^n - (m - 3)^n)m^{-n}$$
$$+ 1((m - 1)^n - (m - 2)^n)m^{-n} = \sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^n,$$

we find $e(n) = \Sigma_k(1 - k^n m^{-n})k/(m - k) \uparrow m\Sigma(1/k) - (m - 1) \sim m \ln m$; for example, $e(n) \uparrow 19\frac{73}{252}$ for $m = 10$. As to e' and q', belonging to the corresponding t-procedure, we have $q' = \Pi_i(1 - (1 - 1/m)^{i^2}) > q$ and, for $m = 2$, $e'(n) = 2^{-n^2}n + (1 - 2^{-n^2})e'(n - 1)$ with $e' \uparrow .5785\ldots$ vs. $e = 1 - 2^{-n}$.

4. Vertex-approach and face-approach methods. Checking all permutation sums of a nonnegative matrix for a least sum can be shortened by dropping all those for which a partial sum surpasses an upper bound, whether otherwise obtained or formed by sums found during the procedure [13].

A more methodic search for matrices consisting of integers [14] can be extended to general matrices as follows. Starting from some permutation p, an element of L_0' is added to a so that all elements in that permutation are 0. If all other elements are $\leqq 0$, p gives a greatest sum; if not, choose a positive element α. Call its column c_1; the row in which c_1 and p meet, r_1; the columns of the nonnegative elements in r_1, $c_1, \cdots, c_{k_1}$; the rows where they and p meet, $r_1, \cdots, r_{k_1}$; the columns of the elements $\geqq 0$ in these rows, $c_1, \cdots, c_{k_2}$; etc. If and when the row of α appears (Fig. 3) a circuit (see Sec. 3(a)) has been found, and hence a greater permutation sum. If not (Fig. 4), we end up with a "tree" of k columns and k rows, $1 \leqq k \leqq n - 1$, such that the $k(n - k)$ elements in the k rows and remaining $n - k$ columns are negative; let β be a greatest of these elements. Subtract β from the k rows and add it to the k columns. If $\alpha + \beta \leqq 0$, the number of positive elements outside p has decreased. If not, the tree has grown. After a finite number of steps we are bound to find a greatest permutation sum. Its unicity can be similarly checked.

Retracing our steps, we see that we have also found an element of L_0' not less than the given matrix and equal to it along a permutation with the greatest sum [2, 3, 15]. This requirement, however, does not in general uniquely determine the element of L_0'.

To try to overcome the combinatorial difficulty of the problem, we may continuize it, making not only the permutation sums $\Sigma_{ij}a_{ij}p_{ij}$ but also their weighted arithmetic means $\Sigma_{ij}a_{ij}\bar{p}_{ij}$, $\bar{p} \in P$, eligible, obviously without affecting the result. We now have a game [6] or linear maximization problem and can use the fact that P has only n^2 faces. Any of the general descent procedures for linear programming [16] can be adapted to our case. From the point of view of the simplex method, the assignment problem is still highly degenerate: every vertex belongs not to $(n - 1)^2$ but to $n^2 - n$ faces and has not $(n - 1)^2$ but

$$\sum_{k=2}^{n} \frac{n!}{(n-k)!k} \sim (n-1)!e$$

adjacent (neighboring) vertices.[2] The method has, though, been worked out for the, more general, transportation problem [17] and seems to be related to the above tree algorithm and to a method for matrices whose elements are integers [18, influenced by 1]. Note that adjacent vertices belong to permuta-

[2] The left divided by the right side attains (just as the entirely unrelated volume of the unit sphere in n-space) its greatest and next-to-greatest value for $n = 5$ and $n = 6$. The left side gives thus the number of independent inequalities $\Sigma a_{ip_i} \geqq \Sigma a_{iq_i}$ that have to hold so that Σa_{ip_i} shall be maximal.

tions obtainable from each other by exchanging along a circuit [19], as we shall see in the next section.

5. The equidistribution problem. Several essential features of the assignment problem are preserved in the following generalization to a rectangular matrix $a = (a_{ij})$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, $m \leqq n$.

We call a nonnegative matrix $\bar{p} = (\bar{p}_{ij})$ an *equidistribution* if its rows have equal sums $\rho > 0$ and also its columns have equal sums $\sigma = \rho m/n$. We consider equidistributions with fixed ρ and σ. Several choices each have their merits: $\rho = 1/m$, $\sigma = 1/n$, $\Sigma_{ij}\bar{p}_{ij} = 1$; $\rho = 1$, $\sigma = m/n$, $\bar{p}$ can be completed by mutually equal elements to a doubly stochastic matrix; $\rho = d/m$, $\sigma = d/n$, $\Sigma_{ij}\bar{p}_{ij} = d = (m,n)$ (the greatest common divisor of m and n) also gives a doubly stochastic matrix for $m = n$; $\rho = n/d$, $\sigma = m/d$ does the same; $\rho = n$, $\sigma = m$.

To find (with given ρ and σ) an equidistribution with greatest $\Sigma_{ij}a_{ij}\bar{p}_{ij}$ is the *equidistribution problem.* We have, as before, an $[(m - 1)(n - 1)]$-space L_1 spanned by the equidistributions and defined by $\Sigma_i a_{ij} = \rho$, $\Sigma_j a_{ij} = \sigma$; its parallel L_0 through 0 is defined by $\Sigma_i a_{ij} = \Sigma_j a_{ij} = 0$. The orthogonal complement, namely, the $(m + n - 1)$-space L_0' consisting of all $(\lambda_i + \mu_j)$, meets L_1 at c, $c_{ij} = \rho/n = \sigma/m$, centroid of the *equidistribution polyhedron* P. That c is the centroid of the vertices of P, of the edges of P, $\cdots$, of P itself, is obvious since in each case the centroid has to be invariant under permutations of the rows as well as under permutations of the columns, and c is the only point on L_1 with this property. The study of the polyhedron P in the general case will throw some light on the square case $m = n$ as well.

The number of faces of P is mn, with two exceptions, since if one hyperplane $a_{ij} = 0$ were to contain no face, none would, and the inequalities are not identically fulfilled on L_1, except for $m = 1$, when $P = L_1$ is a point and has no faces; and since no two of the hyperplanes coincide on L_1, except for $m = n = 2$, when P is a segment and has two faces.

Except for $d = m$ the nonzero elements of a vertex of P are no longer all equal. A vertex of P can be obtained [8] by drawing a diagonal of the m by n rectangle enclosing a and setting $\bar{p}_{ij}$ proportional to the length of the segment of the diagonal in the i,j square (Fig. 5); all other vertices are found from this one by a permutation of the rows and a permutation of the columns. Alternatively divide a segment into m equal parts α_i (in some order) and into n equal parts β_j, and let $\bar{p}_{ij}$ equal the common part of α_i and β_j. It is easy to infer that the number of vertices is $m!n!/(d!l^{n-lm+d}(l - 1)!^m 2^{\epsilon d})$ where $l = [n/m]$ and $\epsilon = 0$ or 1 as $d = m$ or $d < m$. The elements of a vertex are

$$\frac{k\,d\rho}{n} = \frac{k\,d\sigma}{m}, \qquad \left(k = 0, 1, \cdots, \frac{m}{d}\right).$$

The linear subspace defined by $a_{ij} = 0$ for all i and j except $1 \leqq i - km/d \leqq m/d$, $1 \leqq j - kn/d \leqq n/d$, $1 \leqq k \leqq d$, meets P in an $[(m - d)(n - d)/d]$-dimensional *principal face* with $(m/d)!^d(n/d)!^d/(l^{n-lm+d}(l - 1)!^m)$ vertices.

Every vertex belongs to exactly one principal face. P has $m!n!/((m/d)!^d(n/d)!^d d!)$ principal faces. For $d = m$ a principal face is a vertex, for $d = 1$ it is P.

The determination of all edges (segments connecting adjacent vertices) of P is simple for $m = n$. After a permutation of the rows and one of the columns, one of two given vertices may always be supposed to be the identity matrix, while the other consists of one or more circuits. The face hyperplanes common to both contain also every vertex defined by a subset of these circuits. For an edge, that is, for no additional vertex, to arise thus, it is therefore sufficient and necessary that the number of proper circuits (circuits with more than one element) be exactly one. The total number of edges is $\sum_{k=2}^{n} n!^2/((n-k)!2k)$.

If $d = m$, every vertex can be obtained from every other by a permutation of columns. If this permutation (after omitting ineffectual circuits) has more than one proper circuit, again additional vertices belong to the smallest face containing the two vertices. If it has only one proper circuit, say $j_1 \cdots j_k$, and if in the first vertex the corresponding rows $i_1, \cdots, i_k$ (with $\bar{p}_{i_l j_l} \neq 0$) are all different, it is easily seen that no additional vertices exist; on the other hand, if, say, $i_1 = i_h$, then $j_1 \cdots j_k$ may be replaced by the two, not necessarily proper, circuits $j_1 j_{h+1} \cdots j_k$ and $j_2 \cdots j_h$. Hence two vertices are adjacent if and only if one can be obtained from the other by a permutation of the columns with a single proper circuit and such that the rows corresponding, via the nonzero elements of either vertex, to the columns of the circuit are all different. The number of edges from a vertex is $\sum_{k=2}^{m} m!n^k/((m-k)!m^k k)$, altogether $\sum_{k=2}^{m} m!n!n^k/((n/m)!^m(m-k)!m^k 2k)$.

(*a*) Further complications enter into the description of all edges for $m < n$. We can give criteria for a subset of the hyperplanes $a_{ij} = 0$ to contain a face and to be its full defining set; but the application of these criteria and the determination of the dimension of the face are easy only in special cases, for example, for small m and n.

To formulate the criteria first note that an equidistribution cannot be *over-semireducible*, that is, cannot have an "over-half-size grid" of zero elements $\bar{p}_{ij} = 0$, $i = i_1, \cdots, i_\mu$, $j = j, \cdots, j_\nu$, $\mu/m + \nu/n > 1$, for the other elements in the same rows would sum to $\mu\rho > (n - \nu)\sigma$. Secondly, if an equidistribution is *semireducible*, that is, has a "half-size grid" of zeros $\bar{p}_{ij} = 0$, $i = i_1, \cdots, i_\mu, j = j_1, \cdots, j_\nu, \mu/m + \nu/n = 1$, then it is *reducible*, that is, has zeros also at $i \neq i_1, \cdots, i_\mu, j \neq j_1, \cdots, j_\nu$ (in the "complement"), for the elements with $i = i_1, \cdots, i_\mu, j \neq j_1, \cdots, j_\nu$ sum to $\mu\rho = (n - \nu)\sigma$; we say that an equidistribution cannot be *properly semireducible*.

THEOREM 1. *A set of zeros occurs in some equidistribution if and only if it contains no over-half-size grid.*

THEOREM 2. *A set of zeros is the set of all zeros of some equidistribution if and only if it contains no over-half-size grid and with every half-size grid contains its complement.*

COROLLARY. *If $d = m$, then a set of less than mn zeros is the set of all zeros of some equidistribution if and only if, with every half-size grid, it contains its complement.*

The corollary follows from Theorem 2 by remarking that for $d = m$ an over-half-size grid contains a half-size grid and that by twice forming the complements of all half-size grids contained, one gets all mn elements.

To prove Theorem 1 (by double use of the duality, or transposition, theorem for linear inequalities [20]) we examine the relations $\bar{p}_{ij} \geqq 0$, $\Sigma_j \bar{p}_{ij} = \rho$, $\Sigma_i \bar{p}_{ij} = \sigma$ for elements $\bar{p}_{ij}$ outside the given set of zeros. We homogenize to $\bar{p}_{ij} \geqq 0$, $\Sigma_j \bar{p}_{ij} - \rho q = 0$, $\Sigma_i \bar{p}_{ij} - \sigma q = 0$, $q > 0$. By duality, this is insolvable if and only if $\lambda_{ij} - \mu_i - \nu_j = 0$, $\rho \Sigma \mu_i + \sigma \Sigma \nu_j + \kappa = 0$, $\lambda_{ij} \geqq 0$, $\kappa > 0$ is solvable; rewrite as $\mu_i + \nu_j \geqq 0$, $\rho \Sigma \mu_i + \sigma \Sigma \nu_j < 0$. For a solution we may, after rearranging rows and columns, suppose $\mu_1 \leqq \cdots \leqq \mu_m$, $\nu_1 \leqq \cdots \leqq \nu_n$. Then if $\mu_i + \nu_j \geqq 0$, the same is true if i, or j, is increased. We can therefore alter our solution so that μ_1 equals $-\lambda_1$, where λ_1 is the first ν_j relevant to this μ_i (a full row of zeros would be an over-half-size grid); so that the next μ_i to which a smaller ν_j is relevant equals $-\lambda_2$, where $\lambda_2 < \lambda_1$ is the first ν_j relevant to this μ_i; etc. All μ_i between $-\lambda_1$ and $-\lambda_2$ can be set equal to $-\lambda_1$, etc. Now all ν_j after λ_1 can be set equal to λ_1, all those between λ_1 and λ_2 equal to λ_2, etc. We arrive at numbers $\lambda_1 > \cdots > \lambda_k$ such that

$$(\mu_1, \cdots, \mu_m) = (-\lambda_1(\alpha_1 \text{ times}), \cdots, -\lambda_k(\alpha_k \text{ times})),$$
$$(\nu_n, \cdots, \nu_1) = (\lambda_1(\beta_1 \text{ times}), \cdots, \lambda_k(\beta_k \text{ times})),$$

and $\gamma_1\lambda_1 + \cdots + \gamma_k\lambda_k < 0$ where $\gamma_1 = \sigma\beta_1 - \rho\alpha_1$, etc.,

whence $$\gamma_1 + \cdots + \gamma_k = 0.$$

The system $-(\gamma_1\lambda_1 + \cdots + \gamma_k\lambda_k) > 0$, $\lambda_1 - \lambda_2 > 0, \cdots, \lambda_{k-1} - \lambda_k > 0$ is, by duality, solvable if and only if

$$\begin{aligned} -\gamma_1\tau + \tau_1 &= 0, \\ -\gamma_2\tau - \tau_1 + \tau_2 &= 0, \\ -\gamma_3\tau - \tau_2 + \tau_3 &= 0, \\ &\cdots \\ -\gamma_k\tau - \tau_{k-1} &= 0, \\ (\tau, \tau_1, \cdots, \tau_{k-1}) &\geqq (0, \cdots, 0) \end{aligned}$$

is insolvable. We may set

$$\tau = 1, \quad \tau_1 = \gamma_1, \quad \tau_2 = \gamma_1 + \gamma_2, \quad \cdots, \quad \tau_{k-1} = \gamma_1 + \cdots + \gamma_{k-1}.$$

This means that some $\gamma_1 + \cdots + \gamma_l < 0$; but then the first $\alpha_1 + \cdots + \alpha_l$ rows and $n - (\beta_1 + \cdots + \beta_l)$ columns intersect in an over-half-size grid. Q.E.D.

If even no half-size grid occurs in the given set, then

$$\tau_l = \gamma_1 + \cdots + \gamma_l > 0$$

so that $-(\gamma_1\lambda_1 + \cdots + \gamma_k\lambda_k) \geqq 0$, $(\lambda_1 - \lambda_2, \cdots, \lambda_{k-1} - \lambda_k) \geqq (0, \cdots, 0)$ is insolvable. Hence $\mu_i + \nu_j \geqq 0$, $\rho\Sigma\mu_i + \sigma\Sigma\nu_j \leqq 0$ has no solution except $\mu_1 = \cdots = \mu_m$, $\nu_1 = \cdots = \nu_n$ (if, say, the ν_j are unequal, proceed as above; they remain unequal since $k = 1$ implies their equality even before altering). Thus $\lambda_{ij} - \mu_i - \nu_j = 0$, $\rho\Sigma\mu_i + \sigma\Sigma\nu_j + \kappa = 0$, $(\lambda_{ij}, \kappa) \geqq (0, \cdots, 0)$ is insolvable, and there exists a $\bar{p} \in P$ with $\bar{p}_{ij} > 0$ outside the given set. To complete the proof of Theorem 2, note that if the set contains two complementary half-size grids, then the problem reduces to the construction of equidistributions on two disjoint grids on which the suppositions are again fulfilled.

(*b*) As an example, consider the case $m = 2$. Here an over-half-size grid is either a column or more than half of a row. The faces of P are therefore either of type

$$F_j\colon a_{11} = \cdots = a_{ij} = 0,\ j < \frac{n}{2},$$

or

$$F_{jj'}\colon a_{11} = \cdots = a_{ij} = 0,\ a_{2n} = \cdots = a_{2,n-j'+1} = 0,\ j' \leqq j < \frac{n}{2},$$

or

$$F\colon a_{11} = \cdots = a_{1,n/2} = 0,\ a_{2,n/2+1} = \cdots = a_{2n} = 0.$$

Their dimensions are $n - j - 1$, $n - j - j' - 1$, 0. The number f_δ of δ-dimensional faces is thus

$$\binom{n}{\delta + 1} \sum_{k=[n/2-\delta]}^{[(n-1)/2]} \binom{n - \delta - 1}{k}$$

$\left(\text{for } \delta > n/2 - 1 \text{ this is } 2^{n-\delta-1}\binom{n}{\delta+1}\right)$, but $f_0 = \binom{n}{n/2}$ for even n. Setting $f_{-1} = 1$, we obtain the following values for $n \leqq 10$, where a check is provided by $\Sigma(-1)^\delta f_\delta = 0$. $\Sigma f_\delta \sim 3^n$.

n \ δ	−1	0	1	2	3	4	5	6	7	8	9
1	1	1									
2	1	2	1								
3	1	6	6	1							
4	1	6	12	8	1						
5	1	30	60	40	10	1					
6	1	20	90	120	60	12	1				
7	1	140	420	490	280	84	14	1			
8	1	70	560	1120	980	448	112	16	1		
9	1	630	2520	4200	3780	2016	672	144	18	1	
10	1	252	3150	8400	10500	7560	3360	960	180	20	1

To find the volume v of P project on $a_{21} = \cdots = a_{2n} = 0$; the projection cosine is $\gamma = \text{vol } A/\text{vol } (A,A)$, where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ & \ddots & \vdots \\ 0 & 1 & -1 \end{pmatrix} (n - 1 \text{ by } n).$$

Since

$$(\text{vol } A)^2 = |AA'| = \begin{pmatrix} 2 & & 1 \\ & \ddots & \\ 1 & & 2 \end{pmatrix} = 2n - 3$$

and

$$(\text{vol } (A,A))^2 = \begin{pmatrix} 4 & & 2 \\ & \ddots & \\ 2 & & 4 \end{pmatrix} = (2n - 3)2^{n-1},$$

we have $\gamma = 2^{(1-n)/2}$. Denoting by $v_\tau = \tau^n n^{\frac{1}{2}}/n!$ the volume of the simplex $a_{11} \geqq 0, \cdots, a_{1n} \geqq 0, \Sigma a_{ij} = \tau$, a familiar argument shows that

$$v\gamma = v_\rho - nv_{\rho-\sigma} + \binom{n}{2} v_{\rho-2} - + \cdots \left(\frac{n+1}{2} \text{ terms}\right);$$

for example, for $\sigma = 1$ we obtain

$$v = 2^{(n-1)/2} n^{\frac{1}{2}} n!^{-1} \left[\left(\frac{n}{2}\right)^{n-1} - \binom{n}{1}\left(\frac{n}{2} - 1\right)^{n-1} + \binom{n}{2}\left(\frac{n}{2} - 2\right)^{n-1} - + \cdots \right].$$

The projected P is the middle section between opposite vertices of an n-dimensional cube; for a unit cube ($\sigma = 1$) the volume $v\gamma$ is $\sqrt{2}$, $\frac{3}{4}\sqrt{3}$, $\frac{4}{3}$, $\frac{115}{192}\sqrt{5}$, $\frac{11}{20}\sqrt{6}$ for $n = 2, \cdots, 6$. Quite similar but more cumbersome computations will give v for $m > 2$. A general expression, even for $m = n$, is still not known; for $m = n = 3$, $v = \frac{9}{8}$.

Also the values of f_δ are harder to obtain for the $m = n$ sequence of polyhedra. They start:

$n \backslash \delta$	−1	0	1	2	3	4	5	6	7	8	9
1	1	1									
2	1	2	1								
3	1	6	15	18	9	1					
4	1	24	240	978	1968	2176	1392	528	120	16	1

In every case, f_δ as a function of δ has one of the usual bell shapes (which seem to be most lopsided, with a peak at one-third the range, for the cube and its dual).

6. Related problems. We mention briefly various related problems and generalizations.

(1) The transportation problem [8, 18]: To maximize $\Sigma_{ij} a_{ij} \bar{p}_{ij}$ for $\bar{p}_{ij} > 0$, $\Sigma_j \bar{p}_{ij} = \rho_i$, $\Sigma_i \bar{p}_{ij} = \sigma_j$.

(2) The stair problem [21]: To find a greatest $\Sigma a_{i_k j_k}$ with first term a_{11} and last term a_{mn} such that every $(i_{k+1} - i_k, j_{k+1} - j_k)$ is (1,0) or (0,1).

(3) The problem of finding a convex curve, in the sense of Sec. 2(4b) with greatest sum Σa_{ij}.

(4) In a rectangular matrix, to find a greatest sum Σa_{ij} with one term in every row and at most one term in every column.

(5) Or else, with one term in every column and at least one term in every row.

(6) Generalizations of the assignment and the equidistribution problem and of the above five problems to matrices with more than two indices, the candidate sums being taken with one or more indices [20].

(7) Given a matrix q_{jk} of zeros and ones; to find, for given real numbers a_j, a greatest sum $\Sigma a_j q_{jk}$ (see [23]).

(8) Or else, for positive a_j, to maximize $\Sigma a_j p_j$, $p_j > 0$, $\Sigma p_j q_{jk} \leqq 1$, or equivalently to minimize $\max_k \Sigma \lambda_j q_{jk}/a_j$ (generalizing [6] to an arbitrary game matrix whose nonzero elements, in every row, are equal).

Of these, problems 4 and 5 share with the assignment and equidistribution problems the property of invariance under all permutations of rows and of columns. The corresponding polyhedra, and those pertaining to problems 1, 2, and 6, have (for large matrices) much fewer (highest-dimensional) faces than vertices; this suggests that "face-approach" descent methods may work well in these cases.

In problem 4, for $n > m > 1$, the $n!/(n-m)!$ vertices span an $(mn - m)$-dimensional linear subspace, and a polyhedron whose $mn + m$ faces are in the hyperplanes $a_{ij} = 0$, $\Sigma_i a_{ij} = 1$. For positive matrices problem 4 belongs also to the polyhedron $a_{ij} \geqq 0$, $\Sigma_j a_{ij} \leqq 1$, $\Sigma_i a_{ij} \leqq 1$, which arises from a special instance of problem 8. Its vertices are matrices of ones and zeros, with at most one 1 in every row and column. In fact, suppose μ rows and ν columns of a vertex contain elements other than ones and zeros, and let, for example, $\nu \geqq \mu$. Suppose further that $l \geqq 0$ of these columns contain exactly one such element. Then the (μ,ν)-grid concerned contains at least $2\nu - l$ nonzeros. On the other hand, a vertex fulfills some mn of the inequalities as independent equations. Besides the equations $a_{ij} = 0$, any 1 in a vertex gives rise to one equation only, the other being dependent on it and the $a_{ij} = 0$. The (μ,ν)-grid furnishes $\kappa \leqq \mu + \nu - l$ equations other than $a_{ij} = 0$. Thus $\mu + \nu - l \geqq 2\nu - l$, whence $\mu = \nu$, $\kappa = 2\nu - l$. Since the ν rows now sum to ν, so do the ν columns, whence $l = 0$. But the κ equations, together with the $a_{ij} = 0$ outside the grid, are not independent, a contradiction. A third polyhedron equivalent for positive matrices to the former two is $\Sigma_j|a_{ij}| \leqq 1$, $\Sigma_i|a_{ij}| \leqq 1$, with $2^m n!/(n-m)!$ vertices.

To determine the faces and count the vertices in problem 5, let us relax the second condition to "at least one term in l given rows." Necessarily $l \leqq n$, and for $l = n$ we have the assignment problem. For $l < n$ consider, in the $(mn - n)$-dimensional linear subspace $\Sigma_i a_{ij} = 1$, the polyhedron defined by the $mn + l$

inequalities $a_{ij} \geqq 0$, $\Sigma_j a_{ij} \geqq 1$. At a vertex at least $mn - n$ become equations, whence less than $2n$ do not, and we see that not every column contains more than one nonzero. But if we have a column with only one nonzero, we fall back to the same question for $n - 1$. Hence a vertex consists only of zeros and ones, and the polyhedron is the one we were looking for in the first place. The recursion to $n - 1$ gives, for the number $\nu(m,n,l)$ of vertices, the formula $\nu(m,n,l) = (m - l)\nu(m,n - 1,l) + l\nu(m,n - 1,l - 1)$. It is easy to verify the same (and the vanishing for $n = 0$) for

$$m^n - \binom{l}{1}(m-1)^n + \binom{l}{2}(m-2)^n - + \cdots (-1)^l (m-l)^n = \nu(m,n,l).$$

In particular, $\nu(m,n,n) = n!$ and (the vertex number of problem 5)

$$\nu(m,n,m) = m^n - m(m-1)^n + \binom{m}{2}(m-2)^n - + \cdots (-1)^m.$$

Deviating from the assignment problem in direction 7, we can restrict the set of admissible permutations. We mention only sets invariant if the rows and columns undergo the same permutation. Every permutation is thought of as the product of disjoint circuits of $\alpha_1 \geqq \cdots \geqq \alpha_k$ elements, $\Sigma\alpha_1 = n$.

(*A*) The $n!/2$ even permutations (those with even $n - k$).

(*B*) The involutory (or symmetric) permutations (those with $\alpha_1 \leqq 2$).

(*C*) The fixpointless permutations (those with $\alpha_k \geqq 2$).

(*D*) The cycles (permutations with $\alpha_2 = 1$ or $k = 1$).

(*E*) The intersections of these sets, like *BD* (transpositions) and *CD* (full cycles [24]).

The corresponding polyhedra are apt to have many more faces than vertices. For *A*, the f_δ table starts

$n \backslash \delta$	−1	0	1	2	3	4	5	6	7	8	9
2	1	1									
3	1	3	3	1							
4	1	12	66	220	492	768	840	624	288	64	1

The last polyhedron consists of three 4-simplices in three orthogonal 3-spaces which span 9-space. The faces are sixty-four 9-simplices. Every two vertices are adjacent.

Polyhedra with relatively few faces belong to *BD* (a simplex) and *B*. The latter consists of all symmetric doubly stochastic matrices.

Finally, it seems worth while to draw attention (besides to related nonlinear problems [15, 25]) to sequences of doubly stochastic matrices whose limits for $n \to \infty$, if piecewise continuous, might help determine the one-to-one piecewise continuous functions f maximizing $\int F(x,y)\, df$ with given kernel F defined in the unit square, and to a similar variation-theoretic application of problem 4.

7. Geometrical concepts used. (1) Let d, the *dimension*, be a positive integer. A row of d real numbers (*coordinates*) is called a *point* or *vector* in real d-dimensional *space*. The 0 row is called *origin*. Two vectors with vanishing sum of products of corresponding coordinates are *orthogonal*.

(2) For a set S of points, $v \in S$ means the point v is an element of S. The linear combinations $\Sigma\lambda_i v_i$ ($\Sigma\lambda_i = 1$) of finitely many points of S form the *linear subspace L spanned* by S. If L can be spanned by $\delta + 1$ points but not by less, L and S are of *dimension* δ. L can be defined by $d - \delta$ linear equations. For $\delta = d - 1$, L is called *hyperplane*. A *coordinate subspace* is defined by the vanishing of $d - \delta$ coordinates.

(3) Adding one and the same vector to every point of L gives a *parallel* subspace. Two linear subspaces are *orthogonal* if their parallels L_1 and L_2 through the origin are such that any element of L_1 and any element of L_2 are orthogonal. If together they span the whole space, they are *orthogonal complements*.

(4) For a finite set $S = (v_1, \cdots, v_m)$, $\Sigma v_i/m$ is the *centroid* or *arithmetic mean;* $\Sigma\lambda_i v_i$ ($\lambda_i \geqq 0$, $\Sigma\lambda_i = 1$) is a *weighted arithmetic mean*. The weighted arithmetic means of S form a continuum called the *convex hull* $[S]$ of S; $[S]$ is a convex *polyhedron;* $[S]$ can be defined by finitely many linear inequalities.

(5) If the points of a convex polyhedron of dimension δ fulfill a linear inequality, those fulfilling the corresponding equation form a *face* (sometimes "face" stands for "$(\delta - 1)$-dimensional face"). A zero-dimensional face is called a *vertex*, a one-dimensional face an *edge;* an edge connects two *adjacent* or *neighboring* vertices.

(6) For $m = \delta + 1$, $[S]$ is a *simplex*, for $\delta = 1$ a *segment*. The points all of whose coordinates are $\geqq a$ and $\leqq b$ form a *cube* (*unit cube* for $b - a = 1$); two vertices of a cube are *opposite* if no face contains both.

(7) For a simplex $[v_0, \cdots, v_d]$, the *volume* is

$$\frac{|\det(v_1 - v_0, v_2 - v_0, \cdots, v_d - v_0)|}{d!};$$

similarly for a simplex spanning a coordinate subspace. The volume of a polyhedron spanning such a subspace is obtained by decomposition into simplices of its own dimension the common part of any two of which has smaller dimension. For a polyhedron $[S]$ spanning an arbitrary linear subspace L of dimension δ, *project* on every coordinate subspace L_j, $j = 1, 2, \cdots, \binom{d}{\delta}$, of dimension δ (by replacing $d - \delta$ coordinates by 0); then $v^2 = \Sigma v_j^2$, where v_j are the volumes of the projections and $v \geqq 0$ the volume of $[S]$, and the *projection cosine* v_j/v depends only on L and L_j.

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