# GRAPHIC PROGRAMMING USING ODD OR EVEN POINTS

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§1. **Origin of the problem.** When the author was plotting a diagram for a mailman's route, he discovered the following problem: "A mailman has to cover his assigned segment[1)] before returning to the post office. The problem is to find the shortest walking distance for the mailman."

This problem can be reduced to the following: "Given a connected graph in the plane,[2)] we are to draw a continuous graph (repetition permitted) from a given point and back minimizing the number of repeated arcs."

We shall combine the basic concepts concerning continuous graphs and transshipment graphic programming [1] to obtain a method of solution for the above problem, namely graphic programming using odd or even points.

§2. **Some basic concepts.** In order to facilitate our subsequent discussion, we state some basic concepts on linear graphs and known results [2] on continuous graphs relevant to this paper:

1. We decompose the graph into a finite collection of points $P = \{P_1, P_2, \cdots, P_r\}$ and a finite collection of arcs $L = \{l_1, l_2, \cdots, l_s\}$ satisfying the four conditions:

(1) Any $l_k \in L$ contains two distinct endpoints belonging to $P$;

(2) Any $P_i \in P$ is the endpoint of at least one arc in $L$;

(3) Any $P_i$ is not an interior point of any $l_k$;

(4) Any pair of arcs in $L$ cannot have a common interior point.

If $P_i$ is an endpoint of $l_k$, we also say that $l_k$ is an arc emitting from $P_i$. In addition, we also suppose that every arc has a "length," i.e. corresponding to a given real number.

2. Define $l_{i_1}, l_{i_2}, \cdots, l_{i_t}$ to be a chain if their endpoints $(P'_{i_1}, P''_{i_1}), (P'_{i_2}, P''_{i_2}), \cdots, (P'_{i_t}, P''_{i_t})$ satisfy $P''_{i_1} = P'_{i_2}, P''_{i_2} = P'_{i_3}, \cdots, P''_{i_{t-1}} = P'_{i_t}$; but $P'_{i_1} \neq P''_{i_t}$. Call $P'_{i_1}$ and $P''_{i_t}$ the endpoints of this chain. We also say that this chain connects $P'_{i_1}$ and $P''_{i_t}$.

3. A graph is said to be connected if for any two distinct points, there exists a chain connecting them.

4. If the endpoints $(P'_{i_1}, P''_{i_1}), (P'_{i_2}, P''_{i_2}), \cdots, (P'_{i_t}, P''_{i_t})$ of a set of arcs $l_{i_1}, l_{i_2}, \cdots, l_{i_t}$ satisfy $P''_{i_1} = P'_{i_2}, P''_{i_2} = P'_{i_3}, \cdots, P''_{i_t} = P'_{i_1}$, and $P'_{i_1}, P'_{i_2}, \cdots, P'_{i_t}$ are all distinct, then $l_{i_1}, l_{i_2}, \cdots, l_{i_t}$ is a cycle.

5. A point in a graph is called an odd (even) point if the number of arcs emitting from it is odd (even).

6. The number of odd points in any graph is even.

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1) In postal service, a mailman's route is called a segment.

2) A segment is usually connected.

7. A necessary and sufficient condition for the continuous drawing of a connected graph without repetition is that the number of odd points in it be 0 or 2.

8. If continuous drawing is to start from a given point in the graph and back to that point, then a necessary and sufficient condition for a continuous drawing of a connected graph without repetition is that there exists no odd point. When this condition obtains, the graph can be continuously drawn without repetition from any given point in it.

§3. How to minimize redundancy in a graph which can be continuously drawn. If a graph has an odd point, then to require a continuous drawing of the graph from a given point and back must involve repetition. We now study how to minimize the number of repeated arcs. For example, in Diagram 1 the graph has 4 odd points $C, D, E, F$. If we wish to begin with $A$, continuously draw the graph and finally return to $A$, then repetition (of arcs) must occur. If we add all the repeated arcs for a given method of completing the graph, then all points become even. Conversely, if we can first add some arcs so as to change all points in the graph to even points, then the resulting graph may be drawn continuously without repetition, and the added arcs are the repeated ones of the original graph. For example, we can add respectively an arc in Diagram 1 between $C, D$ and $E, F$ and change all points in the graph to even points (see Diagram 2), then we may draw the graph continuously without repetition from $A$ and back to get Diagram 3.

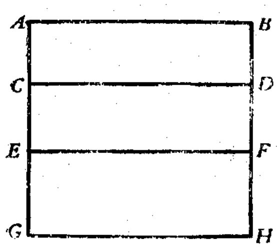

Diagram 1

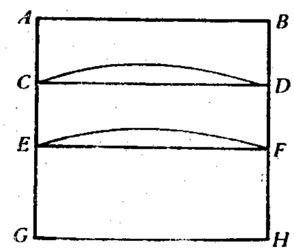

Diagram 2

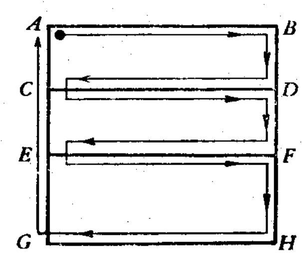

Diagram 3

It follows that the length of the repeated route equals that of the added arcs, and the problem of minimal repeated route in a graph can be reduced to the following: "Suppose a connected graph has $2n$ odd points with all the remaining points even. It is now desired to add "redundant" arcs to some arcs (one arc may have one or more redundant arcs; we suppose the added redundant arc has the same length as the original arc and abbreviate "added redundant arcs" as "added arcs") as to let the resulting graph contain only even points (i.e. the number of added arcs from an odd point must be odd, that from an even point must be even) and also to minimize the total length of the added arcs."

In order to prove the following basic theorem more conveniently, we reformulate this problem in a more general way: "Suppose we designate $2n$ points in a connected graph. We wish to add some arcs to this graph such that the number of added arcs emitting from any of these $2n$ points is odd but that from any of the remaining points is even. Moreover, the total length of the added arcs is to be a minimum."

For convenience, we call the designated $2n$ points odd points and the remaining even points. We call a finite set of added arcs a feasible solution if the number of added arcs from the designated $2n$ points is odd but that from the remaining points is even.

It is not difficult to see that any feasible solution can be regarded as $n$ chains each connecting a pair of odd points. Conversely, any $n$ chains of this sort form a feasible solution. Hence, the existence of a feasible solution is assured. Also, to find completely the set of feasible solutions is rather easy.

Of all feasible solutions the one with the minimum total length is called the optimal solution.

## §4. The fundamental theorem.

**Theorem.** *A sufficient condition that a feasible solution be optimal is that*

(1) *it has no redundancy;*

(2) *the length of added arcs on every cycle does not exceed half the length of the cycle.*

*The redundancy in condition* (1) *refers to the addition of two or more arcs to any arc of the original graph.*

**Proof of necessity.** If a feasible solution fails to satisfy (1), then obviously if we eliminate two redundant added arcs, the remaining added arcs still form a feasible solution with a smaller total length; hence, the original feasible solution cannot be optimal. If a feasible solution violates condition (2), i.e. there exists a cycle with the length of the added arcs exceeding half the length of the cycle, then when this obtains, the total length of the cycle without the added arcs is less than half of the length of the cycle. We now eliminate an added arc from each place in the cycle where there is one originally, and add an arc to each place where there is none originally. Then clearly the total length of added arcs is diminished, and it is not difficult to see that the set of added arcs after the change is still a feasible solution. Hence, the original feasible solution cannot be optimal.

**Proof of sufficiency.** We shall first prove the following lemmas.

**Lemma 1.** *Suppose that two feasible solutions satisfy condition* (1) *and* (2). *Then the lengths of their added arcs are equal.*

**Proof.** Let $M$, $N$ be two feasible solutions satisfying (1) and (2). If $M$ and $N$ both have an added arc to the same arc, we let the endpoints of this arc be $P_i$ and $P_j$. Let us now consider another problem arising from the original problem by interchanging the odd- or evenness of the points $P_i$ and $P_j$ (i.e. arising from interchanging $P_i$ and $P_j$ from "designated" points to "undesignated" points and vice versa). Now we consider two feasible solutions $M'$ and $N'$ of the new problem, $M'$ and $N'$ being obtained respectively from $M$ and $N$ by eliminating the common added arc between $P_i$ and $P_j$. It is easy to see that $M'$ and $N'$ are indeed feasible solutions of the new problem and both still satisfy (1) and (2). Consequently, the proof for the equality of the total lengths of added arcs between $M$ and $N$ is reduced to the proof for that between $M'$ and $N'$. If $M'$ and $N'$ still have added arcs on the same arc, repeat the above elimination process. Therefore, we may assume without loss of generality that $M$ and $N$ originally did not have added arcs on the same arc.

If $M$ and $N$ are both empty, their lengths are clearly equal. Now let $M$ be nonempty. As previously remarked, a feasible solution may be regarded as a set of chains connecting pairs of odd points. We choose any such chain $C_0$ in $M$ with endpoints $P_{i_0}$ and $P_{i_1}$. Since $P_{i_1}$ is odd, there must exist in $N$ a chain $C_1$ with endpoints $P_{i_1}$ and $P_{i_2}$. Choose from $M$ another chain $C_2$ with $P_{i_2}$ as endpoint, $\cdots$. Since the number of odd points is finite, we must obtain, after continuing the process, a chain $C_k$ with endpoints $P_{i_k}$ and $P_{i_{k+1}}$ but $P_{i_{k+1}} = P_{i_j}$, $0 \leq j \leq k-1$. When this obtains, it is not difficult to deduce from the hypothesis that $M$ and $N$ do not both contain added arcs on the same arc, that there exists a cycle in the set of chains $C_j, C_{j+1}, \cdots, C_k$. (If these chains do not intersect themselves, then the cycle is formed from these chains.) Moreover, on every arc of this cycle there exist added arcs belonging to $M$ or $N$.

For the above cycle, the length of added arcs in $M$ plus that in $N$ equals the length of the cycle. Since $M$ and $N$ both satisfy (2), the lengths of added arcs of $M$ and $N$ on this cycle must equal half the length of the cycle.

We can now consider a new problem arising from interchanging the odd- or evenness of the endpoints of the added arcs of $M$ on the above cycle. If a point is the endpoint of two added arcs of $M$, then interchange twice (i.e. there is no change). Consider then the two feasible solutions $M''$ and $N''$ of the new problem. They are obtained from $M$ and $N$ by eliminating all added arcs on the above cycle. It is easy to see that $M''$ and $N''$ indeed are feasible solutions of the problem and still satisfy (1) and (2). Consequently, the proof for the equality of the total lengths of added arcs of $M$ and $N$ can be reduced to that of $M''$ and $N''$.

If $M''$ and $N''$ still are not empty, then further reduction is possible. Since at each step a pair of feasible solutions is reduced to another pair and the number of added arcs is diminished, we must finally reach a pair of feasible solutions which are empty. This shows that $M$ and $N$ must have equal length of added arcs. This completes the proof of Lemma 1.

**Lemma 2.** *Optimal solutions always exist.*

**Proof.** It is not difficult to see that from any feasible solution not satisfying (1) we may eliminate redundancy to obtain another feasible solution with smaller total length and satisfying (1). Moreover, it is also easy to see that the number of feasible solutions satisfying (1) is finite. Hence, among the finite number of feasible solutions satisfying (1), the one with the smallest total length of added arcs has to be optimal. Q.E.D.

By means of Lemmas 1 and 2, the sufficiency part of the theorem is easily proved. In fact, optimal solutions always exist and must satisfy (1) and (2). But all feasible solutions satisfying (1) and (2) have equal total length of added arcs. Hence, all feasible solutions satisfying (1) and (2) must be optimal. This completes the proof of the theorem.

Using the theorem we may now find all optimal solutions. We may first take any feasible solution. If it does not satisfy (1) or (2), then we proceed to adjust it as in the proof of neccessity in the theorem. Since the number of feasible solutions satisfying (1) is finite, and the total length of added arcs is diminished for each adjustment, we must reach an optimal solution after a finite number of steps.

## §5. Steps for finding the optimal mailman's route.

To summarize the above conclusions, we list completely the steps required for finding the optimal mailman's route:

1. Plot a graph of the segment: That is, plot all segments which have to be covered in a diagram. Sometimes mail delivery on two sides of a broad street has to be separately undertaken. Then two arcs must be drawn.

2. Find all odd points in the graph.

3. Use the method discussed above to find an optimal solution, and draw the added arcs of the optimal solution in the graph.

4. Continuously draw without repetition the graph with the arcs added.

**Example.** In Diagram 4, $x$ represents the post office, and we suppose the length of each arc to be proportional to that in the diagram. An odd point is marked with $\circ$; dotted lines indicate added arcs in the optimal solution. If we continuously draw Diagram 4 without repetition, we obtain the optimal mailman's route as in Diagram 5.

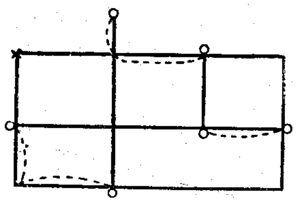

Diagram 4

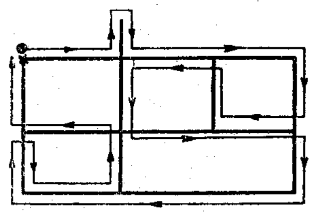

Diagram 5

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[1] *New method for commodity transshipment–Graphic programming*, Mathematical Monthly 1958, No. 11. (Chinese)

[2] Sun Tse-yin, *Introduction to mathematical methods*, Scientific and Technical Publishing Company, 1956. (Chinese)

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