Synthesis of Correct-by-construction Control Protocols for Hybrid Systems Using Partial State Information

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***Abstract*— This paper considers the problem of synthesizing output-feedback control laws for a class of discrete-time hybrid systems in order for the trajectories of the system to satisfy certain high-level specifications expressed in linear temporal logic. By leveraging ideas from robust interpretation of temporal logic formulas and bounded-error estimation, we identify a subclass of systems for which it is possible to reduce the problem to a state-feedback form. In particular, we use locally superstable hybrid observers to resolve the partial information at the continuous level. This allows us to use recent results in temporal logic planning to synthesize the desired controllers based on two-player perfect-information games. The overall control architecture consists of a hybrid observer, a high-level switching protocol and a low-level continuous controller. We demonstrate the proposed framework in a case study on designing control protocols for an aircraft air management system.**

I. Introduction

Correct-by-construction controller synthesis for hybrid systems from temporal logic specifications has attracted considerable attention in the past decade. At present, hybrid systems are put to use both in industrial settings and in products such as cars and airplanes [14]. Safety-criticality of such systems creates a need to synthesize controllers that enforce hybrid systems to satisfy certain high-level specifications, e.g., on safety, reliability and performance.

A typical solution for such control synthesis problems is a hierarchical control architecture with several discrete and continuous layers (see, for instance, [9], [7], [10], [20], [11] and references therein). One of the limitations of these approaches is that they rely on availability of the full system state for feedback. However, in many applications of interest, it is not possible to equip the system with a multitude of sensors both for reasons of economy and physical space. Motivated by these limitations, we propose in this paper a framework that can guarantee correctness with limited measurements through output feedback.

Previous work on synthesis with partial state information has mostly focused on the discrete level [3], [15]. Except for some special cases [8], handling the imperfect state information at the discrete level requires a power set construction (i.e., construction of a belief space) that has prohibitive computational complexity. In this paper, we consider partial observability of the continuous state. In order not to experience a complexity blow-up at the discrete level, we deal with the partial state information at the continuous level. We leverage ideas from robust interpretation of temporal logic formulas [6] and bounded-error estimation [13] to develop a framework for synthesizing provably-correct output-feedback control laws for discrete-time piecewise-affine systems. In particular, for a class of systems admitting locally superstable hybrid observers [4], [2], we show that the problem can be reduced to a state-feedback form, which can then be solved using available tools [21].

The rest of the paper is organized as follows. Section II presents some background results. The problem is formally stated and an overview of the solution strategy is given in section III. The main results are presented in sections IV-V. Section VI demonstrates an application of the proposed framework to a case study on aircraft air management systems. Finally, section VII concludes the paper with some remarks.

II. Preliminaries

A. Notation

All vector and matrix norms considered in this article will be the infinity norms, denoted without subscripts. Therefore, given a matrix $A = (A)_{ij}$ and a vector $x = [x_1, \ldots, x_n]^T$, we define $\|A\| = \max_i \sum_j |A_{ij}|$, $\|x\| = \max_i |x_i|$. The i^{th} row vector of A is written as $[A]_i$. The row and column spaces of A is denoted by row(A) and col(A), respectively, with dimensions dim(row(A)) and dim(col(A)); and the dimension of the null space is written as nullity(A). The diameter of a set $X \subseteq R^n$ is denoted by $\mathrm{diam}(X)$, and its distance from a point $p \in \mathbb{R}^n$ by $d(p, X) = \inf_{x \in X} \|p - x\|$.

With a point $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$, we denote the ball centered at p with radius r as $B(p, r) = \{x \in \mathbb{R}^n : \|x - p\| \leq r\}$. Lastly, given a matrix $H \in \mathbb{R}^{m \times n}$ and a vector $k \in \mathbb{R}^m$, a *polytope* is a set $P = \{x : Hx \leq k\} \subseteq \mathbb{R}^n$, where the inequality is interpreted element-wise, i.e., $P = \{x : [Hx]_i \leq [k]_i, i = 1, \ldots, m\}$.

Given a set Q, Q^ω (Q^+) denotes the set of infinite (non-empty finite) sequences of elements in Q.

B. System and environment models

We consider discrete-time piecewise affine systems formally defined as follows.

Definition 1: A discrete-time piecewise-affine system is a tuple $S = (X, \{R_k\}_{k=1}^{k_{max}}, \{\mathfrak{D}_k\}_{k=1}^{k_{max}})$ where:

- $X \subseteq \mathbb{R}^n$ is a compact set called the state space.

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- The regions $R_k \subseteq X$, $R_i \cap R_j = \emptyset$ for $i \neq j$, form a partition of X.
- $\mathfrak{D}_k$ is the dynamics in region R_k, that is the state x evolves with

$$\begin{aligned} x(t+1) &= A_k x(t) + B_k u(t) + F_k + E_k \delta(t) \\ y(t) &= C_k x(t) \end{aligned} \tag{1}$$

  when $x(t) \in R_k \subseteq X$. In Eq. (1), $y(t) \in \mathbb{R}^l$ is the measured output, $u(t) \in U \subseteq \mathbb{R}^m$ is the control input, $x(t) \in \mathbb{R}^n$ is the state variable, and $\delta(t) \in W \subseteq \mathbb{R}^d$ is the disturbance. $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m}$, $C_k \in \mathbb{R}^{n \times l}$, $F_k \in \mathbb{R}^{n \times 1}$ and $E_k \in \mathbb{R}^{n \times d}$ are the system matrices corresponding to region R_k.

We let Y denote the set of outputs, that is, $Y \doteq \{Cx : x \in X\}$. Given a system S, a subset $X' \subseteq X$ is said to *respect the dynamics* if $X' \cap R_k \neq \emptyset$ only for a unique k.

In addition to the disturbances in the system model, we consider an external *environment* to refer to the "discrete" factors that are relevant to the operation of the system, but do not impact its dynamics directly, i.e., not explicitly appear in Eq. (1). Since such factors are not necessarily controlled by the system, e.g., traffic lights, weather conditions, user inputs etc., they are treated as adversaries. We use a simple transition system to model environment evolution.

Definition 2: An environment model is a tuple $\mathcal{T}_e = (\mathcal{E}, \mathcal{E}_0, \rightarrow)$ where:

- $\mathcal{E}$ is a finite set of states.
- $\mathcal{E}_0 \subseteq \mathcal{E}$ is a set of initial states, i.e., $e(0) \in \mathcal{E}_0$.
- $\rightarrow \subseteq \mathcal{E} \times \mathcal{E}$ is a transition relation that governs the evolution of the environment. That is, $(e(t), e(t+1)) \in \rightarrow$ for all $t \geq 0$.

The discrete environment is assumed to be fully observable by the system.

Remark 1: For the clarity of the presentation, we restrict the system dynamics to the form in Eq. (1). Our framework can be easily extended to cases where there is measurement noise or where the dynamics include controllable and uncontrollable switches (e.g., using ideas from [16], [11]). Also, our framework allows general U, W and $\{R_k\}_{k=1}^{k_{max}}$ sets, but, in what follows, we assume these sets are bounded convex polytopes to facilitate computation.

C. Linear temporal logic and protocols

Linear temporal logic (LTL) is a formal language that extends the standard propositional logic with temporal operators to express complex, temporal tasks [1]. LTL has proven useful in e.g., software and hardware verification, robotics and other applications of control synthesis, allowing for succinct and expressive specification of system behavior.

1) Syntax and semantics: Before defining the syntax and semantics of LTL, we need a few definitions. A combined state of the system and the environment is a tuple $s(t) \doteq (e(t), x(t)) \in \mathcal{E} \times X$ and a *trajectory* is an infinite sequence of states of the form $s = s(0)s(1)\ldots \in (\mathcal{E} \times X)^\omega$. An *atomic proposition* is a function from the set of states to boolean $true$ and $false$. We denote the set of all atomic propositions by Π. In our context each $\pi_i \in \Pi$ is an indicator function of a set $[\![\pi_i]\!] = \{e\} \times X_i$, with $e \in \mathcal{E}$ and $X_i \subseteq X$ is a convex polytope, wherein π evaluates to $true$. A set $X' \subseteq X$ is said to *respect the propositions* if for all $x \in X'$ the same set of propositions hold.

The syntax of an LTL formula over a set of atomic propositions Π is given by the following grammar:

$$\varphi ::= true | \pi | \varphi_1 \wedge \varphi_2 | \neg\varphi | \bigcirc \varphi | \varphi_1 \mathcal{U} \varphi_2,$$

where $\pi \in \Pi$, and φ_1 and φ_2 are LTL formulas. The symbols $\wedge$, $\neg$, $\bigcirc$ and $\mathcal{U}$ stand for the logical operators conjunction, negation and temporal operators next and until, respectively. These operators can be used to define additional operators such as disjunction ($\vee$), implication ($\rightarrow$), always ($\Box$) and eventually ($\Diamond$). We will consider specifications in an assume/guarantee form

$$\varphi \doteq \varphi_e \rightarrow \varphi_s, \tag{2}$$

where φ_e encodes assumptions on the environment and φ_s specifications of the desired system behavior.

Satisfaction of a formula φ at a state $s(t)$ in a trajectory s is denoted by $s(t) \models \varphi$ and is defined by letting

1) $s(t) \models true$;
2) For any atomic proposition π, $s(t) \models \pi$ if $s(t) \in [\![\pi]\!]$;
3) $s(t) \models \varphi_1 \wedge \varphi_2$ if $s(t) \models \varphi_1$ and $s(t) \models \varphi_2$;
4) $s(t) \models \neg\varphi$ if $s(t) \not\models \varphi$;
5) $s(t) \models \bigcirc\varphi$ if $s(t+1) \models \varphi$;
6) $s(t) \models \varphi_1 \mathcal{U} \varphi_2$ if $\exists j \in \mathbb{N}$ s.t. $s(i) \models \varphi_1$, $s(j) \models \varphi_2$, $\forall i \in [t, j)$.

The trajectory s satisfies the formula φ if $s(0) \models \varphi$. We say that a formula φ of the form (2) is *satisfied by the system* if it is satisfied by all possible trajectories of the system which are consistent with the dynamics in (1) and for all environment behaviors captured by the environment model.

A *state-feedback control protocol* $\mathcal{C}$ is a partial function on non-empty sequences of states of the system with

$$\begin{aligned} \mathcal{C} : (\mathcal{E} \times X)^+ \times \mathcal{E} &\rightarrow U \\ (s(0), s(1) \ldots, s(t-1), e(t)) &\mapsto u(t) \end{aligned} \tag{3}$$

where $u(t)$ is the input signal to be used in the subsequent time-step. Lastly, by letting $r(t) \doteq (y(t), e(t))$, we define an *output-feedback control protocol* as

$$\begin{aligned} \mathcal{C} : (\mathcal{E} \times Y)^+ \times \mathcal{E} &\rightarrow U \\ (r(0), r(1) \ldots, r(t-1), e(t)) &\mapsto u(t) \end{aligned} \tag{4}$$

where $u(t)$ is decided upon by only using the measured output.

2) Robust satisfaction of LTL formulas: Following [6], this section describes a robust interpretation of LTL formulas. Given $\pi \in \Pi$, define π^ε, by $[\![\pi^\varepsilon]\!] = \{(e, x) \in [\![\pi]\!] : (e, x') \in [\![\pi]\!], \forall x' \in B(x, \varepsilon)\}$. $[\![\pi]\!]$ denotes a robust version of the atomic proposition $[\![\pi]\!]$, which will be used in connection with estimation errors below. We can extend the robustness properties to general LTL formulas as follows. Take φ as a formula written on Negation Normal Form [5]. Form $\neg\Pi = \{\neg\pi : \pi \in \Pi\}$ and let $\hat{\Pi} = \Pi \cup \neg\Pi$. Now, interpreting φ as a formula over $\hat{\Pi}$, replace all atomic propositions π by

π^ε and denote the resulting formula by φ^ε. We say that a system *satisfies a formula* φ *ε-robustly* if it satisfies φ^ε.

III. Problem formulation and solution strategy

Next, we formally state the problem and give an overview of the proposed solution.

Problem 1: Given a system $S = (X, \{R_k\}_{k=1}^{k_{max}}, \{\mathfrak{D}_k\}_{k=1}^{k_{max}})$, an environment $\mathcal{T}_e = (\mathcal{E}, \mathcal{E}_0, \rightarrow)$, a set Π of propositions together with an LTL formula φ as in (2), and a set $X_0 \subseteq X$ that respects the propositions and dynamics, construct an output feedback control protocol that satisfies φ for all initial conditions $x(0)$ in X_0 and for all possible environment behaviors in $\mathcal{T}_e$ using only the measured output y.

We denote an instance of Problem 1 as a tuple $(S, X_0, \mathcal{T}_e, \varphi)$. When a solution to a problem $(S, X_0, \mathcal{T}_e, \varphi)$ exists, we say the problem is *realizable*.

Starting from a system model given in the form of Def. 1 and an LTL specification in the form of (2), we use an approach centered on observer estimations of the state space in order to solve Problem 1. The proposed framework exploits the hierarchical approach considered in earlier work [20], [16] and consists of the following steps:

1) Find a locally superstable observer with an appropriate equalized performance level and redefine the system dynamics and LTL specifications based on the estimated state.
2) Produce a discrete abstraction based on the redefined dynamics.
3) Use existing techniques in automata theory to design a control protocol guaranteeing correctness of the system.
4) Implement the automaton for continuous execution by combining the observer with low-level controllers.

In section IV, we discuss certain types of observers that are suitable for steps 1 and 4. In section V, we prove that given such an observer, one can still guarantee correctness when using the redefined dynamics in steps 2 and 3 and treating the problem as a state-feedback problem as in [20], [16]. We also briefly overview the results from [20], [16] necessary to complete these design steps.

The overall control architecture shown in Fig. 1 consists of a hybrid observer, a high-level switching protocol and a low-level continuous controller.

IV. Observer design

In order to utilize possible partially known state information, we use observers

$$\begin{aligned} \mathcal{O} : (Y \times U)^* &\rightarrow X \\ (p(0), p(1) \ldots, p(t-1)) &\mapsto \hat{x}(t) \end{aligned} \tag{5}$$

where $p(t) \doteq (y(t), u(t))$ and $\hat{x}(t)$ is an estimate of the state. The estimation error at time t is denoted by $\xi(t) \doteq x(t) - \hat{x}(t)$. The design of an observer is made more difficult for piecewise-affine systems as an error in the estimate ambiguities the underlying dynamics. Moreover, any atomic proposition $\pi \in \Pi$ holds true in a limited region $[\![\pi]\!] \subseteq \mathcal{E} \times X$; and $(e, \hat{x}) \in [\![\pi]\!]$ does not imply $(e, x) \in [\![\pi]\!]$. Therefore, upper bounds on the estimation errors are needed.

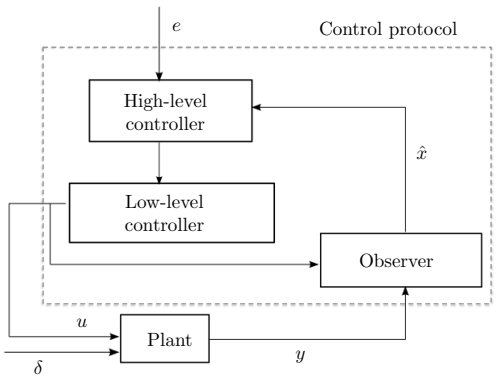

Fig. 1. The proposed control architecture.

Typically, optimal observers that minimize the estimation error when there are persistent disturbances can be arbitrarily complex even for linear systems [18], [13]. Instead of seeking optimal bounds, we adopt the notion of equalized performance from [2] to characterize observers.

Definition 3: [Equalized performance] An observer is said to achieve an *equalized performance level* μ if, whenever $\|\xi(t)\| \leq \mu$, we have $\|\xi(t+1)\| \leq \mu$.

For a piecewise-affine system as in Def. 1, we consider fixed-complexity locally-affine observers of the form

$$\hat{x}(t+1) = (A_k - L_k C_k)\hat{x}(t) + B_k u(t) + F_k + L_k y(t), \tag{6}$$

with a collection of linear filter gains $L_k \in \mathbb{R}^{n \times l}$, one for each R_k.

Proposition 1: Consider an observer of the form (6). Assume, for the time being, that the observer has perfect knowledge of k (i.e., the region R_k the true state $x(t)$ is in) at all times.[1] Then, choosing the filter gains L_k in Eq. (6) such that

$$\|A_k - L_k C_k\| \leq 1 - \frac{\max_{\delta \in W} \|E_k \delta\|}{\varepsilon} \tag{7}$$

for all k leads to an equalized performance level ε.

Proof: Since k is known by assumption, the estimation error evolves as

$$\xi(t+1) = (A_k - L_k C_k)\xi(t) + E_k \delta(t). \tag{8}$$

By equating the norms of both sides of Eq. (8) and with simple manipulation, one can see that if $\|\xi(t)\| \leq \varepsilon$ and Eq. (7) holds, we have $\|\xi(t+1)\| \leq \varepsilon$. ■

Definition 4: [Locally detectable system] An observer of the form (6) that satisfies Eq. (7) is called a *locally superstable observer with equalized performance* ε. A piecewise affine system is called *locally detectable with performance level* ε if it admits a locally superstable observer with equalized performance ε.

[1]Note that this is true at $t = 0$ because X_0 respects the dynamics. In section V, we show how to synthesize the control protocol so that it chooses the consequent control inputs u to ensure this at later time steps.

Note that this condition is more restrictive than the notion of detectability as, firstly, detectability only concerns the behavior of a trajectory as time goes to infinity, and secondly, the existence of a matrix L such that $\|A - LC\| < 1$ as in Eq. (7) is a sufficient condition for detectability of (A, C).

Finally note that checking if a system is locally superstable with a given performance level and if so, finding the corresponding filter gains, can be efficiently performed via linear programming.

A. Conditions on system matrices for superstability

In this section, we present an alternative characterization of a subclass of systems for which there exist locally superstable observers with equalized performance ε. These characterizations can provide guidelines for designing the measurement matrices C_k, when sensor selection is part of the design. Proofs of the following propositions can be found in [12].

Proposition 2: There exists a family of filter gains L_k for a system of the form (1) with $\|A_k - L_k C_k\| \leq \varepsilon$ if and only if for each k, there exists a $D_k \in \mathbb{R}^{n \times n}$ with

1) $\text{row}(A_k + D_k) \subseteq \text{row}(C_k)$
2) $\|D_k\| < \varepsilon$.

Note that as $\dim(\text{col}(C_k^T)) \leq l$, a filter achieving the required bound is more easily constructed when $A_k + D_k$ has low rank or when $l \to n$, which intuitively corresponds to the two cases where either $A_k + D_k$ does not hold much information or when C_k gives practically full state information, respectively.

Proposition 3: If there exists a family of filter gains L_k for a system of the form (1) with $\|A_k - L_k C_k\| \leq \varepsilon$, then for each k, there exists a $D_k \in \mathbb{R}^{n \times n}$ with

1) $\text{nullity}(A_k^T + D_k^T) \geq n - l$
2) $\|D_k\| < \varepsilon$.

Note that this sufficiently characterizes the systems for which the developed framework is possible to use.

V. Reduction to state feedback

Obviously, if we can ensure that the estimated state trajectory robustly satisfies an LTL formula and the estimation error can be kept globally bounded, then the true trajectory satisfies the LTL formula. This fact is stated formally next.

Theorem 1: Let $s = s(0)s(1)...$ be an infinite sequence where $s(t) = (e(t), x(t))$ for all t. Similarly, define $\hat{s} = \hat{s}(0)\hat{s}(1)...$ where $\hat{s}(t) = (e(t), \hat{x}(t))$ for all t. Given an LTL formula φ, if there exists a bound $\varepsilon \geq 0$ such that $\hat{s} \models \varphi^{\varepsilon}$ and $||\hat{x}(t) - x(t)|| \leq \varepsilon$ for all t, then $s \models \varphi$.

Proof: Follows directly from the definition of robust satisfaction of LTL formulas. ∎

In order to be able to employ this result, we need to establish a global bound ε on the estimation error using locally superstable observers. We first consider an intermediate result.

Lemma 1: Let $S = (X, \{R_k\}_{k=1}^{k_{max}}, \{\mathfrak{D}_k\}_{k=1}^{k_{max}})$ be a system. If S is locally detectable with performance level $\text{diam}(X_0)$ and $\hat{x}(t) \in \bigcup_{k=1}^{k_{max}} \hat{R}_k$ for all $t \geq 0$, where $\hat{R}_k \doteq \{x \in R_k : B(x, \text{diam}(X_0)) \subseteq R_k\}$ then $\|\xi(t)\| \leq \text{diam}(X_0)$ for all $t \geq 0$.[2]

Proof: Since S is locally detectable with performance level $\text{diam}(X_0)$, there exist L_k such that Eq. (7) holds. By Proposition 1, the estimation error can be bounded if the unique region R_k with $x \in R_k$ is known. By the assumptions in problem formulation 1, this is known at $t = 0$ and we proceed by induction on t. Given $x(t) \in R_k$, we know $\|\xi(t)\| \leq \text{diam}(X_0)$. By the definition of the shrunk regions, $d(x(t), \hat{R}_j) > \text{diam}(X_0)$, for $j \neq k$ so if $\hat{x}(t) \notin \hat{R}_k$, then $\|\xi(t)\| = \|x(t) - \hat{x}(t)\| \geq d(x(t), \hat{R}_j) > \text{diam}(X_0)$, which contradicts the induction hypothesis. Therefore, $\hat{x}(t) \in \hat{R}_k$ and, in the next time step, we can measure $\hat{x}(t+1) \in \hat{R}_j$, for some $1 \leq j \leq k_{max}$ and obtain $\|\xi(t+1)\| \leq \text{diam}(X_0)$, by Proposition 1. This concludes the induction step and the proof is done. ∎

We associate each locally detectable system with another system whose outputs are equal to its states.

Definition 5: Given a locally detectable system $S = (X, \{R_k\}_{k=1}^{k_{max}}, \{\mathfrak{D}_k\}_{k=1}^{k_{max}})$ that admits a locally superstable observer with performance level ε and corresponding gains L_k, the *ε-robust observer system* $\hat{S} = (\hat{X}, \{\hat{R}_k\}_{k=1}^{k_{max}}, \{\hat{\mathfrak{D}}_k\}_{k=1}^{k_{max}})$ is given by the following parameters:

- $\hat{X} = \bigcup_{k=1}^{k_{max}} \hat{R}_k$,
- $\hat{R}_k = \{x \in R_k : B(x, \varepsilon) \subseteq R_k\}$,
- $\hat{\mathfrak{D}}_k$ is the dynamics in region $\hat{R}_k$, with

$$\begin{aligned} \hat{x}(t+1) &= \hat{A}_k \hat{x}(t) + \hat{B}_k \hat{u}(t) + \hat{F}_k + \hat{E}_k \hat{\delta}(t) \\ \hat{y}(t) &= \hat{x}(t), \end{aligned} \tag{9}$$

where $\hat{A}_k = A_k$, $\hat{B}_k = B_k$, $\hat{F}_k = F_k$, $\hat{E}_k = L_k C_k$, $\hat{u}(t) \in \hat{U} = U$ and $\hat{\delta}(t) \in \hat{W} = B(0, \varepsilon)$.

An observer for S as in Eq. (6) is said to be *consistent* with an ε-robust observer system $\hat{S}$ if they use the same gains L_k.

Now we state the main result of this section where ε-robust observer systems are used to pose an alternative perfect-information problem, the solution of which provides a solution to Problem 1.

Theorem 2: Define $\varepsilon' \doteq \text{diam}(X_0)$. Given an instance $(S, X_0, \mathcal{T}_e, \varphi)$ of Problem 1, assume S is locally detectable with performance level ε'. Let $\hat{S} = (\hat{X}, \{\hat{R}_k\}_{k=1}^{k_{max}}, \{\hat{\mathfrak{D}}_k\}_{k=1}^{k_{max}})$ be a ε'-robust observer system for S. Then, if there exists a state-feedback control protocol for $\hat{S}$ that makes the system satisfy $\varphi^{\varepsilon'}$ for some initial condition $\hat{x}(0) \in X_0 \cap \hat{X}$ and for all possible environment behaviors in $\mathcal{T}_e$, $(S, X_0, \mathcal{T}_e, \varphi)$ is realizable. Moreover, an output-feedback protocol for S can be constructed by using an observer consistent with the dynamics of $\hat{S}$ and by driving the system S with the same control signals as applied to $\hat{S}$.

Proof: By definition, the state-feedback control protocol for $\hat{S}$ that makes the system satisfy $\varphi^{\varepsilon'}$ ensures that $\hat{x}(t) \in \hat{X}$ for all t, i.e., the states remain within the state-space of the system. Therefore, by Lemma 1, the estimation

[2] To be precise, a performance level strictly less than $\text{diam}(X_0)$ is required to accommodate trajectories with $x(t)$ on a border between two regions. However, such cases are negligible from a practical standpoint and will be disregarded.

error can be globally bounded by ε' for any initial condition $\hat{x}(0) \in X_0 \cap \hat{X}$ while running the system S and the observer with the input signal from this protocol. Finally, invoking Theorem 1 concludes the proof. ■

A. Overview of full-information synthesis

This section briefly describes the process of obtaining state-feedback control protocols. The full details can be found in e.g., [16], [20].

Based on previous work [16], [20], a discrete synthesis procedure can be phrased as a two-player perfect information game, where the environment is treated as an adversary, i.e., it is assumed to make the worst-case transitions consistent with its transition relation in Def. 2 and the assumption φ_e part of the specification. In order to incorporate the piecewise affine system in this game, constructing a finite transition system representing the dynamics is required [20]. This construction relies on partitioning the continuous state-space to create discrete-states and solving short-horizon constrained reachability problems between regions to establish the transition relations.

In order to solve the state-feedback synthesis problem stated in Theorem 2, we create a discrete-transition system for the ε-robust observer system, where the reachability computations are performed on shrunk regions. The first step in doing so produces a proposition preserving partition $X = \bigcup_{i=1}^{n} P_i$ respecting the system dynamics. Assuming P_i to be a convex polytope defined by $H_i x \leq k_i$, a shrunk polytope can then be defined as

$$\hat{H}_i x \leq \hat{k}_i, \tag{10}$$

where $\hat{H}_i = H_i$ and $[\hat{k}_i]_j = [k_i]_j - \varepsilon \|[H_i]_j\|$. This gives the following result.

Proposition 4: If $\|\xi(t)\| \leq \varepsilon$ and $\hat{x}(t) \in \hat{P}_i$, then $x(t) \in P_i$.

Proof: We have $H_i x(t) = H_i \left(x(t) - \hat{x}(t) + \hat{x}(t)\right) = H_i \left(x(t) - \hat{x}(t)\right) + \hat{H}_i \hat{x}(t)$. In the last equality, $\|x(t) - \hat{x}(t)\| \leq \varepsilon$, and by construction, $[\hat{H}_i \hat{x}(t)]_j \leq [k_i]_j - \varepsilon \|[H_i]_j\|$, so we obtain $[H_i x(t)]_j \leq [\hat{H}_i \hat{x}(t)]_j + \varepsilon \|[H_i]_j\| \leq [k_i]_i - \varepsilon \|[H_i]_i\| + \varepsilon \|[H_i]_j\| = [k_i]_j$. ■

Now, appealing to Theorem 2, transitions for the estimated state $\hat{x}$ between regions $\hat{P}_i$ and $\hat{P}_j$ lead to transitions of the actual state between regions P_i and P_j, meaning that algorithms for control synthesis [21], [17] applied directly to the estimated system in Definition 5 lead to control protocols of the original system in Problem 1.

Lastly, note that if a region P_i in Proposition 4 is given by a union of several convex polytopes, shrinking each of these will yield a valid, although conservative, region for control synthesis.

We summarize the ideas above in the following algorithm:

1) Establish a proposition preserving partition $X = \bigcup_{i=1}^{n} P_i$ of the state space domain X, respecting the system dynamics.
2) Shrink each region P_i. Establish transition relations for each of the new polytopes $\hat{P}_i$.
3) Force $\hat{x}$ to transition between the shrunk polytopes.

VI. Case study: Air management system of aircraft

This section uses a simplified and linearized model of an air management system (AMS) of an aircraft as a test-case for the theory developed in sections II-V above.

A conventional AMS operates by admitting ambient air into the engines of an aircraft and forwarding this to a so called pressurization and air conditioning kit, where pressure is controlled by electrical compressors, temperature by a heat exchanger (HX) and possibly expansion cooling in a turbine, and finally humidity by a high pressure water extraction loop [14]. The AMS needs to be designed so as to supply sufficient pressure to the cabin at bearable temperature and humidity, preferably under comfortable conditions. It is also responsible for providing the cabin with its supply of fresh oxygen. Restrictions in the amount of power that can be supplied to the electronics and sensitivity of sensors to e.g, high temperature, exist; also, freezing of different parts of the craft pose operational problems. Lastly, the AMS should be fault tolerant as it is a critical part of the craft.

TABLE I

The symbols used in the simplified AMS model.

Symbol	Unit	Description
Measurable states		
T_x	° C	Temperature of metal in heat exchanger
T_c	° C	Temperature of cabin
Non-measurable state		
p_v	kPa	Outlet air pressure of valve 1
Controllable variables		
C_1		Valve coefficient for valve 1
C_2		Valve coefficient for valve 2
W_a	kg/s	Mass flow rate of cold inflow in HX
Switched variables		
T_a	K	Temperature of cold inflow in HX (ambient air)
T_e	K	Temperature of the air from the engine
Other derived variables		
W_i	kg/s	Incoming mass flow rate of the air from the engine
W_v	kg/s	Mass flow rate of the air that goes through valve 2
W_h	kg/s	Mass flow rate of the air that goes through the HX
T_h	K	Outlet air temperature of the HX
Constant variables		
p_e	kPa	Pressure of the air from the engine
p_c	kPa	Pressure of the cabin
W_f	kg/s	Mass flow rate passing through the fan

TABLE II

Numerical values used in the simplified AMS model

Symbol	Value	Description
Measurable states		
T_x	297.2 K	Equilibrium value
T_c	268 K	Equilibrium value
Non-measurable state		
p_v	136791 Pa	Equilibrium value
Controllable variables		
C_1	0.155	Equilibrium value
C_2	0.18	Equilibrium value
W_a	2.49 kg/s	Equilibrium value
Switched variables		
T_a	-39, 161 ° C	Arbitrary value
T_e	207, 25 ° C	Arbitrary value
Constant variables		
p_e	275.790 kPa	Arbitrary value
p_c	101.325 kPa	Arbitrary value

A simplified schematic of an AMS is included in Figure VI and the details of the model can be seen in [12]. The symbols used in this section are given in Table I with numerical

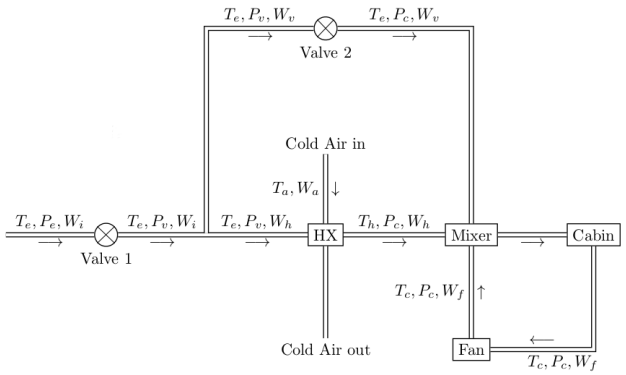

Fig. 2. A simplified AMS model. Variables in this figure are defined in Table I.

values listed in Table II. The units of the parameters are suppressed in the text below. We introduce switching to the system by assuming the engine temperature to toggle uncontrollably between $T_e = 207$ and $T_e = 25$. Also, the airplane can switch its dynamics by having ambient air at either $T_a = -39$ or heated to $T_a = 161$. The uncontrollable switches are assumed to have a time scale larger than the sampling time of the controller, and is here set to 0.5 seconds.

We consider the state space $X = \{[T_c, T_x, p_v]^T \in \mathbb{R}^3 : 13 \leq T_c \leq 33,\ -25 \leq T_x \leq 15,\ 101.325 \leq p_v \leq 275.790\}$. Due to the non-linear dynamics of the system, the model results in a set of piecewise affine and linearized dynamics, with three different regions of definition, for every choice of the two environmental and controllable switching modes. These are determined by $R_1 = \{[T_c, T_x, p_v]^T \in X : 101.325 \leq p_v \leq 137.895\}$, $R_2 = \{[T_c, T_x, p_v]^T \in X : 137.895 \leq p_v \leq 202.65\}$ and $R_3 = \{[T_c, T_x, p_v]^T \in X : 202.65 \leq p_v \leq 275.790\}$, respectively. The control inputs are given by $U = \{[C_1, C_2, W_a]^T \subseteq \mathbb{R}^3 : 0 \leq C_1, C_2 \leq 1,\ 0 \leq W_a \leq 8.316\}$. Lastly, in order to obtain interesting results with limited hardware, the B-matrices obtained are amplified by a factor of 7.5. In all, this gives a discrete-time switched piecewise affine system with dynamics of the form

$$\begin{aligned} \begin{bmatrix} T_c(t+\Delta t) \\ T_x(t+\Delta t) \\ p_v(t+\Delta t) \end{bmatrix} &= A_k \begin{bmatrix} T_c(t) \\ T_x(t) \\ p_v(t) \end{bmatrix} + B_k \begin{bmatrix} C_1(t) \\ C_2(t) \\ W_a(t) \end{bmatrix} + \\ &\quad + F_k + E_k \delta(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_c(t) \\ T_x(t) \\ p_v(t) \end{bmatrix}, \end{aligned} \tag{11}$$

for $k = 1, 2, 3$. We use a sampling time $\Delta t = 0.1$ seconds.

A. Specifications

We assume the cabin crew to be able to set the reference values of T_c to hot ($T_c \in I_1 = [23.5, 25]$), cold ($T_c \in I_2 = [21, 22.5]$) or intermediate ($T_c \in I_3 = [22.5, 23.5]$). Cabin crew input is treated as the environment $\mathcal{E}$. The system should eventually reach the reference levels and stay within these levels until told otherwise and we require the environment to not change the reference value until the reference interval has been reached. Also, the cabin temperature should always stay within the temperature range $T_c \in [21, 25]$. Lastly, we require to always have non-freezing heat exchanger temperature in order to prevent freezing. In order to phrase this in LTL, we represent the cabin crew reference value by a level variable $l \in \{1, 2, 3\}$ which corresponds to when the reference value is hot, cold and intermediate, respectively. We also introduce a timer $t \in \{0, 1, \ldots, 5\}$ and require the reference values to be constant when $t \neq 5$ in order to increase the time scale of the reference value change. The specifications then become:

$$\begin{aligned} &\varphi_e \rightarrow \varphi_s, \\ &\varphi_e = \left(\bigwedge_{i=1}^{3} \square \left((l = i \wedge T_c \in I_i) \rightarrow \bigcirc (l) = i \right) \right) \wedge \\ &\quad \wedge \square \left((t \neq 5) \rightarrow (\bigcirc (t) = t + 1) \right) \wedge \\ &\quad \wedge \square \left((t = 5) \rightarrow (\bigcirc (t) = 0) \right) \wedge \\ &\quad \wedge \square \left((t \neq 5) \rightarrow (\bigcirc (l) = l) \right), \\ &\varphi_s = \left(\bigwedge_{i=1}^{3} \square (l = i \rightarrow \Diamond T_c \in I_i) \right) \wedge \\ &\quad \wedge \left(\bigwedge_{i=1}^{3} \square ((l = i \wedge T_c \in I_i) \rightarrow \bigcirc (T_c) \in I_i) \right) \wedge \\ &\quad \wedge \square \Diamond (T_x \geq 0). \end{aligned} \tag{12}$$

The controllers were synthesized using the Temporal Logic Planning (TuLiP) Toolbox [21], which is a software package designed for temporal logic motion planning interfacing with JTLV [17]. The techniques considered in this article are, however, not limited to this particular choice of software.

B. Simulation

A sample simulation is included in Figures 3-4 below, where the initial error in T_c and T_x were 0.0175 and the initial error in p_v was 1.75. The disturbance term was bounded by 0.016 and 1.6 for T_c, T_x and p_v, respectively. For these values and the numerical values of the system matrices, Propositions 2 and 3 can be seen to guarantee existence of a locally superstable observer. With the time t in minutes, the simulation runs with the reference temperature set to intermediate for $0 \leq t < 4$, $16 \leq t < 20$, hot for $4 \leq t < 10$ and cold for $10 \leq t < 16$. In the figures, note the reference following of the cabin temperature and that T_c and p_v always remain within the state space. The error magnitudes never exceed their initial values, due to local superstability and reduce to the magnitude of the disturbance term during the simulation. Numerically, $\|T_c(t) - \hat{T}_c(t)\| \leq 0.016$, $\|T_x(t) - \hat{T}_x(t)\| \leq 0.016$, $\|p_v(t) - \hat{p}_v(t)\| \leq 1.6$ for all times. Note that T_c has an error term due to the effect of the disturbance for which the observer cannot compensate between a time step and the next.

VII. Conclusions and Future Research

In this paper, we described a framework for synthesizing correct-by-construction control protocols for discrete-time

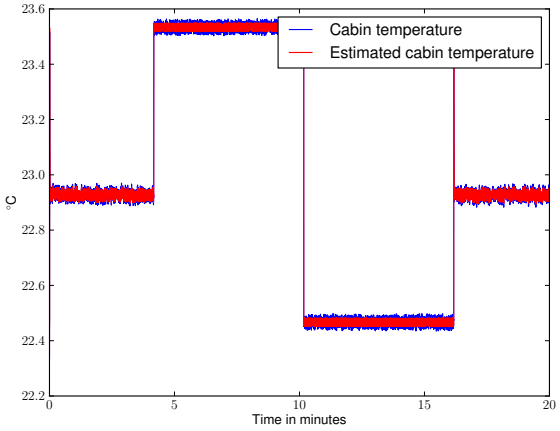

Fig. 3. Cabin temperature for the sample AMS simulation

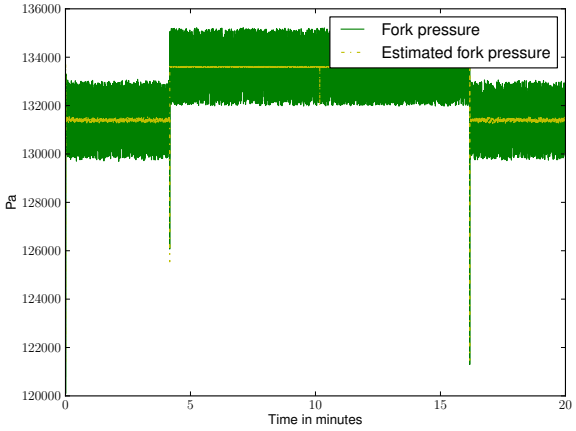

Fig. 4. Pipe fork pressure for the sample AMS simulation

piecewise-affine systems using only partial state information. The main insight of the proposed approach is to resolve the uncertainty in the continuous state at the continuous level of a hierarchical controller so that it is possible to solve a full information problem at discrete level. Ideas from robust estimation were used to design appropriate local observers that are synergistically integrated with the controller stack to achieve global bounds on the estimation errors. The approach was demonstrated on a case-study in the form of an air management system of aircraft.

Future research will consider employing nonlinear or higher-order observers within the proposed framework. Another interesting direction is to investigate whether the relation between the dynamics of a system and the corresponding bounded-error observers can be characterized in terms of alternating approximate simulation relations [19].

Acknowledgment

This work was supported in part by IBM and UTC through the iCyPhy consortium. The authors wish to thank the iCyPhy team, in particular Yilin Mo from Caltech and Jeff Ernst from UTAS, for helpful discussions on the AMS example and modeling.

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