On the Non-Existence of Immersions for Systems with Multiple Omega-Limit Sets ⋆

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Abstract: Linear immersions (or Koopman eigenmappings) of a nonlinear system have wide applications in prediction and control. In this work, we study the existence of one-to-one linear immersions for nonlinear systems with multiple omega-limit sets. For this class of systems, existing work shows that a discontinuous one-to-one linear immersion may exist, but it is unclear if a continuous one-to-one linear immersion exists. Under mild conditions, we prove that systems with multiple omega-limit sets cannot admit a continuous one-to-one immersion to a class of systems including linear systems. Multiple examples are studied to verify our results.

Keywords: Nonlinear Systems, Immersion, Koopman Operator, Equivalent Systems

1. INTRODUCTION

Finding linear approximations (or linearization) of nonlinear systems is one important topic in the analysis and control of nonlinear systems. A classical linearization technique is to take the first-order Taylor expansion of the system at an equilibrium as a linear approximation. The resulting linear system reflects the local behaviors of the nonlinear system near the equilibrium, such as stability of the equilibrium (Hirsch et al. (2012)), but is less useful when the states leave the neighborhood of the equilibrium. The Koopman operator theory provides a way to find global linearizations of nonlinear systems. Specifically, if there exists a finite-dimensional one-to-one Koopman eigenmapping defined on a forward invariant set of the states (Lan and Mezić (2013)), the nonlinear system in this forward invariant set can be immersed in a finite-dimensional linear system. This linear system is called a Koopman representation of the nonlinear system. Compared with the local linearization by Taylor expansion, the Koopman-based linearization captures the global behavior of the system (Mauroy et al. (2020); Brunton et al. (2021)) and thus has great potential in applications involving prediction and control. Recent works show promising results of applying Koopman representations in model reduction and control of PDEs (Kutz et al. (2018); Peitz and Klus (2020)), prediction of chaotic systems (Brunton et al. (2017)), modeling and control of soft robots (Bruder et al. (2020)), and model predictive control of nonlinear systems (Korda and Mezić (2018)).

The idea behind Koopman representations and embeddings of nonlinear systems in linear (or bilinear, when there are controls) systems has appeared often in the control literature, although under different names. Finite-dimensional embeddings correspond to finite-dimensional spaces of observables (Wang and Sontag (1989)). The Koopman representation can be interpreted as the "dual system" used in linear theory (Kalman duality) and more generally as the foundation of the duality between observability of a nonlinear system and controllability of a (generally infinite dimensional) system of observables, the *adjoint system*. See for example the work in Sontag and Rouchaleau (1976); Sontag (1979, 1995) on algebraic observability (strong reachability of the adjoint system, and subjective comorphisms into cosystems in the first reference) and a brief mention in Exercise 6.2.10 in the textbook Sontag (1998). A very closely related concept, but for infinite-dimensional linear systems, is the "topological observability", which amounts to the exact reachability of a dual system (Yamamoto (1981)).

In this work, we also call a finite-dimensional Koopman eigenmapping a *linear immersion*. The aforementioned applications of Koopman-based linearization require knowing a one-to-one linear immersion explicitly so that the state of the original system can be recovered from the state of the Koopman representation. Numerical methods are developed to find approximations of such one-to-one linear immersions, for instance, by searching over a heuristic collection of observables (Brunton et al. (2016, 2021)). However, a one-to-one linear immersion may not exist for arbitrary systems. When it does not exist, numerical methods may have trouble finding a good approximation. So it is important to study when a one-to-one linear immersion does not exist. In this work, we focus on the existence of one-to-one linear immersions for systems with multiple ω-limit sets. Specifically, our main contributions include:

- Under mild conditions, we prove that nonlinear systems with more than one ω-limit set cannot admit a

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continuous one-to-one immersion to a class of systems, including linear systems.
- We demonstrate our theoretical results with multiple examples.

We believe our results can shed some light on when it may or may not be a good idea to use Koopman liftings in practice.

Related work: A key difference between linear and nonlinear systems is that nonlinear systems can have multiple isolated ω-limit sets while linear systems cannot. Thus, there are many debates in the literature on whether a system with more than one isolated ω-limit set can be immersed in a linear system. In particular, Brunton et al. (2016) claim that a system with multiple isolated ω-limit sets cannot have a linear immersion with a linear inverse. (A linear immersion has a linear inverse if the original state variables are equal to linear combinations of the states of the immersion.) This claim is supported by Williams et al. (2015), which observes that the leading Koopman eigenfunctions of a specific system with three equilibria have support only in one of the basins of attraction. However, Bakker et al. (2019) disproves that claim of Brunton et al. (2016) by providing a counterexample. In their example, a one-dimensional system with three isolated equilibria admits a linear immersion with a linear inverse. It is worth noting that this linear immersion is discontinuous at the boundary of two basins of attraction, which is consistent with our results. Our results show that if this example has a one-to-one linear immersion, it must be discontinuous (since we prove that no continuous one exists). Bakker et al. (2019, 2020) further provide some theoretical analysis that helps explain why the linear immersion may be discontinuous at the boundary of basins of attraction. However, their analysis only considers one type of ω-limit set, that is equilibrium. Our results hold for any type of ω-limit sets.

The paper is organized as follows: In Section 2, we introduce some preliminaries on dynamical systems and immersions. Our main theoretical results are stated and then demonstrated with multiple examples in Section 3, and the proof of the main results is presented in Section 4. We conclude the paper in Section 5.

Notation: We denote the closure of set X by $cl(X)$. The symbols $\mathbb{R}$ and $\mathbb{R}^+$ denote the real line and the set of non-negative real numbers. The symbol $\mathbb{Z}$ denotes the set of integers.

2. PRELIMINARIES

We consider a continuous-time autonomous system defined on a manifold $\mathcal{M}$:

$$\dot{x} = f(x), \ x \in \mathcal{M}. \tag{1}$$

Given an initial state $\xi \in \mathcal{M}$, we denote the solution of the system in (1) by $\varphi(t, \xi)$, with $\varphi : \mathbb{R}^+ \times \mathcal{M} \to \mathcal{M}$. That is, $\varphi(0, \xi) = \xi$ and

$$\frac{d\varphi(t,\xi)}{dt} = f(\varphi(t,\xi)). \tag{2}$$

Let $\mathcal{X}$ be a forward invariant subset of the manifold $\mathcal{M}$ that represents the region in which we want to analyze the system behavior. We endow $\mathcal{X}$ with the subspace topology induced from $\mathcal{M}$. Throughout the paper, we will assume that the system in (1) is forward complete, and the function $f(\cdot)$ in (1) is smooth enough to guarantee the existence and uniqueness of $\varphi(t, \xi)$ for any initial state ξ in $\mathcal{X}$. In addition, we make the following assumption.

Assumption 1. The set $\mathcal{X}$ is path-connected.

Given an initial state ξ of a system in (1), we denote the ω*-limit set* of ξ in $\mathcal{X}$ by $\omega^+(\xi)$, that is the set of all $x \in \mathcal{X}$ such that there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \varphi(t_n, \xi) = x$ (Hirsch et al. (2012)).

Definition 1. Given an initial state $\xi \in \mathcal{X}$, the trajectory $\varphi(t, \xi)$ is called *precompact* in $\mathcal{X}$ if the closure of the set $\{\varphi(t, \xi) \mid t \geq 0\}$ is sequentially compact with respect to the subspace topology on $\mathcal{X}$.

The following lemma states sufficient (and necessary) conditions for the nonemptiness of $\omega^+(\xi)$.

Lemma 1. For any $\xi \in \mathcal{X}$, the limit set $\omega^+(\xi)$ is nonempty if the trajectory $\varphi(t, \xi)$ is precompact in $\mathcal{X}$. If the system is linear with $\mathcal{M} = \mathbb{R}^n$, then the converse is also true. □

Proof. The forward implication is well known. We recall the standard proof here. Suppose $\varphi(t, \xi)$ is precompact in $\mathcal{X}$. Let t_n be a sequence such that $t_n \to \infty$. By Definition 1, there exists a subsequence t_{n_k} such that $\varphi(t_{n_k}, \xi)$ converges to a point x in the closure of $\{\varphi(t, \xi) \mid t \in \mathbb{R}^+\}$. Thus, $\omega^+(\xi)$ contains x and is nonempty.

Now suppose that the system in (1) is linear, that is, $\dot{x} = Ax$ for some $A \in \mathbb{R}^{n\times n}$. If a solution $\varphi(t, \xi)$ of the linear system is unbounded, it can be shown that $\lim_{t\to\infty} \|\varphi(t, \xi)\|_2 = \infty$. Then, $\omega^+(\xi)$ is empty since for any sequence $t_n \to \infty$, $\|\varphi(t_n, \xi)\|_2 \to \infty$. Thus, if $\omega^+(\xi)$ is nonempty, $\varphi(t, \xi)$ is bounded and thus is precompact in $\mathbb{R}^n$. Since $\omega^+(\xi)$ is in $\mathcal{X}$, the closure of $\varphi(t, \xi)$ in $\mathbb{R}^n$ is contained in $\mathcal{X}$, which implies that $\varphi(t, \xi)$ is precompact in $\mathcal{X}$. □

Definition 2. Let $\mathcal{W}$ be the set of all possible ω-limit sets of $\dot{x} = f(x)$ in $\mathcal{X}$. For each $\Omega \in \mathcal{W}$, we define its *domain of attraction* by

$$D(\Omega) = \{\xi \in \mathcal{X} \mid \omega^+(\xi) = \Omega\}. \tag{3}$$

By definition, the set $\mathcal{W}$ contains all the equilibria or closed orbits in $\mathcal{X}$. Next, we introduce the definition of immersion, which generalizes the notion of Koopman eigenfunctions (Mauroy et al. (2020)).

Definition 3. A system $\dot{x} = f(x)$ on $\mathcal{X} \subseteq \mathcal{M}$ is *immersed* in a system $\dot{z} = g(z)$ on a manifold $\mathcal{Z}$ if there is a continuous mapping $F : \mathcal{X} \to \mathcal{Z}$ (an immersion) such that, for all initial states $\xi \in \mathcal{X}$ and all time $t \geq 0$,

$$F(\varphi(t, \xi)) = \psi(t, F(\xi)),$$

where $\psi(t, F(\xi))$ is the solution of $\dot{z} = g(z)$.

If the system $\dot{z} = g(z)$ above is linear, the mapping F is called a *linear immersion*.

Remark 1. We directly require an immersion F in Definition 3 to be continuous for simplicity, since the remainder of this paper only deals with the existence of continuous immersions. In general, an immersion does not have to be continuous. □

Remark 2. Linear immersions are tightly related to Koopman operator theory (Brunton et al. (2016)): A *Koopman eigenfunction* F is a linear immersion that immerses $\dot{x} = f(x)$ in a one-dimensional system $\dot{z} = \lambda z$ for some

$\lambda \in \mathbb{R}$. The span of the entries of any linear immersion F is *a Koopman invariant subspace.* □

Remark 3. If an immersion F is one-to-one, the inverse $F^{-1} : F(\mathcal{X}) \to \mathcal{X}$ exists. Thus we can retrieve the solution $\varphi(t, \xi)$ of $\dot{x} = f(x)$ for any $\xi \in \mathcal{X}$ from the solution $\psi(t, F(\xi))$ of $\dot{z} = g(z)$ via the formula $\varphi(t, \xi) = F^{-1}(\psi(t, F(\xi)))$. □

Remark 4. The term "immersion" is also widely used in the study of differentiable manifolds (see for example Lee (2012)), which is unrelated to the immersion of dynamical systems considered in this work. □

For a given system $\dot{x} = f(x)$, we are most interested in figuring out the existence of a one-to-one linear immersion F, since in this case the behaviors of the nonlinear system $\dot{x} = f(x)$ are fully captured by a finite-dimensional linear system $\dot{z} = Az$, whose behaviors are much easier to study.

Finally, we introduce a class of systems with a special property of the domain of attraction $D(\Omega)$. This class includes all the linear systems. Later we show that this special property is the main reason why a one-to-one linear immersion may not exist for a system with more than one ω-limit sets.

Recall that $\mathcal{W}$ is the set of all ω-limit sets in $\mathcal{X}$.

Definition 4. A system of the form (1) has *closed basins* if the domain of attraction $D(\Omega)$ is closed for all ω-limit sets $\Omega \in \mathcal{W}$.

Lemma 2. Every linear system $\dot{x} = Ax$ (with $\mathcal{X} = \mathbb{R}^n$) has closed basins. □

Proof. Let x_0 be a limit point of $D(\Omega)$ for some $\Omega \in \mathcal{W}$. Denote the span of $D(\Omega)$ by S. Since S contains $D(\Omega)$ and is closed, $x_0 \in S$. By the superposition property of linear systems, since $D(\Omega)$ is forward invariant, S is also forward invariant. Thus, without loss of generality, we can restrict the state space of the system to S.

By Lemma 1, all trajectories in $D(\Omega)$ are precompact. Then, by the superposition property again, all trajectories with initial states in the span S of $D(\Omega)$ are precompact. That implies the system restricted to S is stable in the sense of Lyapunov. Thus, there exists $M > 0$ such that $\|\exp(At)x\|_2 \leq M\|x\|_2$ for all $x \in S$ and $t \geq 0$. Since x_0 is a limit point of $D(\Omega)$, there exists a sequence x_k in $D(\Omega)$ such that $x_k \to x_0$. Then, for all $k > 0$, for all $t \geq 0$, since $x_0 - x_k \in S$,

$$\begin{aligned}\|\varphi(t, x_0) - \varphi(t, x_k)\|_2 &= \|\exp(At)(x_0 - x_k)\|_2 \\ &\leq M\|x_0 - x_k\|_2 \xrightarrow{k\to\infty} 0. \end{aligned} \quad (4)$$

Note that (4) implies that $\omega(x_0) = \omega(x_k) = \Omega$. Thus, $x_0 \in D(\Omega)$, that is, $D(\Omega)$ is closed. □

3. MAIN THEOREM

The following theorem states our main results:

Theorem 1. Under Assumption 1, suppose that:

(T1) $\dot{x} = f(x)$ on $\mathcal{X}$ can be immersed in a system with closed basins by a one-to-one mapping F;
(T2) trajectories of $\dot{x} = f(x)$ on $\mathcal{X}$ are precompact in $\mathcal{X}$;
(T3) the set $\mathcal{W}$ is finite or countable.

Then $\mathcal{W}$ has exactly one element.

Combining Lemma 2 and Theorem 1, the following corollary states a necessary condition for the existence of a one-to-one linear immersion (or in general a one-to-one immersion to systems with closed basins).

Corollary 3. Suppose $\mathcal{X}$ contains at most countably many ω-limit sets of $\dot{x} = f(x)$. Then $\dot{x} = f(x)$ on $\mathcal{X}$ can be immersed in a linear system (or a system with closed basins) by a one-to-one immersion F only if $\mathcal{X}$ contains at most one ω-limit set, or $\mathcal{X}$ contains an unbounded trajectory of the system. □

Next, we verify our results in Theorem 1 and Corollary 3 with several examples. For each example, we first show that a one-to-one (linear) immersion of the given system can be constructed when $\mathcal{X}$ satisfies the conditions in Corollary 3; and then, we show that the constructed immersion becomes discontinuous or ill-defined when we modify $\mathcal{X}$ slightly to violate one of those conditions.

Example 1. Consider the 1-dimensional system

$$\dot{x} = x^2 - 1. \quad (5)$$

The ω-limit sets of the system are $\{-1\}$ and $\{1\}$. Let $\mathcal{X} = (-\infty, 1)$, which only contains one ω-limit set $\{-1\}$. It can be shown that $\dot{x} = x^2 - 1$ on $\mathcal{X}$ is immersed in $\dot{z} = -2z$ by the one-to-one mapping

$$F(x) = \frac{x+1}{x-1}. \quad (6)$$

However, if we extend $\mathcal{X}$ by a point to $\mathcal{X}' = (-\infty, 1]$, the function F in (6) is not an immersion anymore, since $F(1)$ is not defined. This observation is explained by Corollary 3: Since $\mathcal{X}'$ contains two limit sets $\{-1\}$ and $\{+1\}$, and all trajectories in $(-\infty, 1]$ are precompact, there does not exist a one-to-one linear immersion for the system on $\mathcal{X}'$. □

Example 2. Consider the 1-dimensional system:

$$\dot{x} = \sin(x). \quad (7)$$

Let $\mathcal{X} = [0, \pi]$. The ω-limit sets of the system are $\{0\}$ and $\{\pi\}$. Define $y = \cos(x)$. Then, the derivative of y satisfies

$$\dot{y} = -\sin(x)^2 = \cos(x)^2 - 1 = y^2 - 1, \quad (8)$$

with $|y| \leq 1$. That is, the system in (7) on $\mathcal{X}$ is immersed in the system in (5) on $\mathcal{Z} = [-1, 1]$. In this example, $\mathcal{W}$ has two elements, and all trajectories of x in $\mathbb{R}$ are precompact, but a one-to-one immersion exists. By Theorem 1, this is possible only if the system $\dot{y} = y^2 - 1$ does not have closed basins. Indeed, the domain of attraction $D(\{-1\})$ of the system of y on $\mathcal{Z}$ is $[-1, 1)$, not a closed set.

Furthermore, by using Example 1, the system of y on $[-1, 1)$ can be immersed in $\dot{z} = -2z$ with the immersion in (6). Thus, $\dot{x} = \sin(x)$ on $\mathcal{X}' = (0, \pi]$ is immersed in $\dot{z} = -2z$ with the one-to-one mapping

$$F(x) = \frac{\cos(x) + 1}{\cos(x) - 1}. \quad (9)$$

If we extend $\mathcal{X}'$ to $\mathcal{X} = [0, \pi]$, the function $F(x)$ in (9) is not defined at 0 and thus is not an immersion on the closed interval. This can be again explained by Corollary 3 since all the trajectories of x are precompact, and the interval $[0, \pi]$ contains two limit sets. □

Example 3. Consider the one-dimensional system

$$\dot{x} = x - x^3. \quad (10)$$

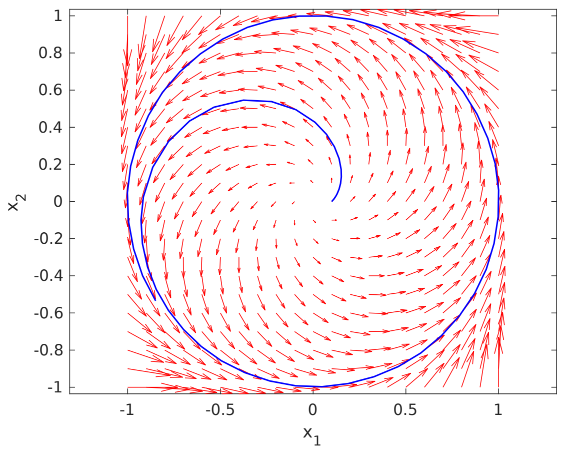

Fig. 1. The phase portrait (red) of the system in (12). The blue curve shows a trajectory of the system that starts from $(0.1, 0)$ and converges to the unit circle.

The ω-limit sets of the system are $\{-1\}$, $\{0\}$ and $\{1\}$. Let $\mathcal{X} = \mathbb{R}\backslash\{0\}$. Define $y = x^{-2} - 1$. Then, $\dot{y}$ satisfies

$$\dot{y} = -2x^{-3}(x - x^3) = -2y. \tag{11}$$

Thus, the system in (10) on $\mathcal{X}$ is immersed in $\dot{y} = -2y$ with the immersion $F(x) = x^{-2} - 1$. Similar to the previous example, $\mathcal{X}$ contains two ω-limit sets, but each of its path-connected components contains only one ω-limit set and thus the result is consistent with Corollary 3. □

Example 4. Consider a 2-dimensional system

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{12}$$

with state $x = (x_1, x_2)$. The ω-limit sets of the system are the origin $\{0\}$ and the unit circle $\{x \mid \|x\|_2 = 1\}$, which can be be seen from the phase portrait in Fig. 1.

Let $\mathcal{X} = \mathbb{R}^2\backslash\{0\}$. Define a function $F : \mathcal{X} \to \mathbb{R}^3$ by

$$F(x) = (x_1/\|x\|_2, x_2/\|x\|_2, \|x\|_2). \tag{13}$$

For a solution $x(t)$ of the system in (12), it can be checked that $F(x(t))$ is a solution of the following system:

$$\begin{aligned} \dot{u} &= -v, \\ \dot{v} &= u, \\ \dot{r} &= r - r^3. \end{aligned} \tag{14}$$

Thus, the system in (12) on $\mathcal{X}$ is immersed in the system in (14) with the immersion F in (13). Using Example 3, the dynamics of r coordinate on $(0, +\infty)$ can be immersed in $\dot{w} = -2w$ with $F(r) = r^{-2} - 1$. Thus, the system of x in (12) on $\mathcal{X}$ is immersed in the following linear system

$$\begin{aligned} \dot{u} &= -v, \\ \dot{v} &= u, \\ \dot{w} &= -2w, \end{aligned} \tag{15}$$

with the one-to-one mapping

$$F(x) = (x_1/\|x\|_2, x_2/\|x\|_2, \|x\|_2^{-2} - 1). \tag{16}$$

The linear immersion $F(x)$ is continuous in $\mathcal{X}$. But if we extend $F(x)$ from $\mathcal{X} = \mathbb{R}^2\backslash\{0\}$ to $\mathbb{R}^2$, $F(x)$ is discontinuous at the origin and thus is not an immersion anymore. This can be explained by Corollary 3: Since all trajectories of x are precompact and the set $\mathbb{R}^2$ contains two ω-limit sets, there does not exist a one-to-one linear immersion for the system of x on $\mathbb{R}^2$. □

4. PROOF OF THE MAIN THEOREM

To prove Theorem 1, we first need to introduce two lemmas. The first lemma reveals a relation between ω-limit sets of the original system and the immersed system.

Lemma 4. Let F be an immersion that maps $\dot{x} = f(x)$ on $\mathcal{X}$ to $\dot{z} = g(z)$ on $\mathcal{Z}$. For each $\xi \in \mathcal{X}$ with a trajectory precompact in $\mathcal{X}$, $F(\omega^+(\xi)) = \omega^+(F(\xi))$.

Proof. We first prove that $F(\omega^+(\xi)) \subseteq \omega^+(F(\xi))$. Indeed, suppose that $p \in \omega^+(\xi)$, and pick a sequence of times $t_i \to \infty$ so that $\varphi(t_i, \xi) \to p$ as $t_i \to \infty$. Therefore $\psi(t_i, F(\xi)) = F(\varphi(t_i, \xi)) \to q := F(p)$, showing that $q \in \omega^+(F(\xi))$.

Conversely, suppose that $q' \in \omega^+(F(\xi))$, and pick a sequence of times $t_i \to \infty$ so that $F(\varphi(t_i, \xi)) = \psi(t_i, F(\xi)) \to q' \in \mathcal{Z}$ as $t_i \to \infty$. Since the trajectory $\varphi(t, \xi)$ is precompact in $\mathcal{X}$, there is a subsequence of the t_i's, which is again denoted by t_i without loss of generality, so that $\varphi(t_i, \xi) \to p \in \mathcal{X}$ and thus $\psi(t_i, F(\xi)) = F(\varphi(t_i, \xi)) \to q := F(p)$. Since we picked a subsequence, also $\psi(t_i, F(\xi)) = F(\varphi(t_i, \xi)) \to q'$. We conclude that $q' = q \in F(\omega^+(\xi))$, showing that $\omega^+(F(\xi)) \subseteq F(\omega^+(\xi))$. We conclude that $F(\omega^+(\xi)) = \omega^+(F(\xi))$. □

Remark 5. Under condition (T2) in Theorem 1, for any $\Omega \in \mathcal{W}$, the image $\hat{\Omega} := F(\Omega)$ is an ω-limit set for the system $\dot{z} = g(z)$. Indeed, by definition there is some $\xi \in \mathcal{X}$ such that $\omega^+(\xi) = \Omega$. Thus, from Lemma 4 we have that $\hat{\Omega} = F(\Omega) = F(\omega^+(\xi)) = \omega^+(F(\xi))$. □

Next, observe that, in general, $F(D(\Omega)) \neq D(F(\Omega))$, since the latter set could be larger. Examples are easy to construct by taking $\mathcal{X}$ to be a forward-invariant subset of $\mathcal{Z}$ and F the identity. For example, consider $\dot{x} = -x$ on $\mathcal{X} = (-1, 1)$ and the same system $\dot{z} = -z$ on $\mathcal{Z} = \mathbb{R}$. Here $\Omega = \{0\}$ is the only ω-limit set, and $F(D(\Omega)) = D(\Omega) = (-1, 1)$ but $D(F(\Omega)) = \mathbb{R}$. However, the following weaker statement is true.

Lemma 5. Suppose that F is an immersion and every trajectory of the system $\dot{x} = f(x)$ on $\mathcal{X}$ is precompact in $\mathcal{X}$. Then, $D(\Omega) \subseteq F^{-1}(D(F(\Omega)))$ for each $\Omega \in \mathcal{W}$. Moreover, if F is one-to-one, then $D(\Omega) = F^{-1}(D(F(\Omega)))$.

Proof. Pick any limit set $\Omega \in \mathcal{W}$ for the system $\dot{x} = f(x)$ and consider $\hat{\Omega} := F(\Omega)$.

Take any $\xi \in D(\Omega)$. This means that $\omega^+(\xi) = \Omega$, and therefore, since 4 $\hat{\Omega} = F(\Omega) = F(\omega^+(\xi)) = \omega^+(F(\xi))$, we conclude that $F(\xi) \in D(\hat{\Omega})$. It follows that $\xi \in F^{-1}(D(\hat{\Omega}))$.

Now suppose that F is one-to-one. Conversely, pick any $\xi \in F^{-1}(D(\hat{\Omega}))$, which means that $F(\xi) \in D(\hat{\Omega})$. Let $\Omega_0 := \omega^+(\xi)$ (well-defined because every trajectory is precompact in $\mathcal{X}$), so $\xi \in D(\Omega_0)$. Thus, $F(\Omega_0) = F(\omega^+(\xi)) = \omega^+(F(\xi)) = \hat{\Omega}$. Since $F(\Omega) = \hat{\Omega} = F(\Omega_0)$ and F is one-to-one, we conclude that $\Omega = \Omega_0$, and thus $\xi \in D(\Omega_0) = D(\Omega)$. Since ξ was arbitrary, we conclude $F^{-1}(\hat{\Omega}) \subseteq D(\Omega)$, as desired. □

Proof of Theorem 1: Since by (T2) every trajectory is precompact in $\mathcal{X}$ and by (T3) there are at most countably many ω-limit sets in $\mathcal{X}$, we have that $\mathcal{X} = \bigcup_{i \in I} D(\Omega_i)$, for a finite or countable set I. These sets are disjoint.

They are also closed, because $D(\Omega_i) = F^{-1}(D(F(\Omega_i)))$ by Lemma 5 and (T1), and F is continuous, and $D(F(\Omega_i))$ is closed. So $\mathcal{X}$ is a disjoint union of a countable collection of closed sets, and by Assumption 1, it is a path-connected topological space. This means that only one of these sets can be nonempty, by a theorem of Sierpiński (1918). □

Remark 6. Sierpiński's Theorem states that if a continuum $\mathcal{X}$ has a countable cover $\{X_i\}_{i=1}^{\infty}$ by pairwise disjoint closed subsets, then at most one of the sets X_i is non-empty. A continuum is a compact connected Hausdorff space, but we do not assume that $\mathcal{X}$ is compact. However, the theorem is still true if $\mathcal{X}$ is not compact. Indeed, suppose that two of the sets X_i would be nonempty, and pick two points p, q, one in each set. Consider a (continuous) path $\gamma : [0, 1] \to \mathcal{X}$ that joins these two points, and let $\Gamma := \gamma([0, 1])$. Now the sets $\{X_i \bigcap \Gamma\}_{i=1}^{\infty}$ form a disjoint cover of the continuum Γ, but two of these sets are nonempty, a contradiction. □

5. CONCLUSION

In this work, we show that a system with multiple ω-limit sets cannot admit a one-to-one immersion to a linear system (or a system with closed basins) under the assumptions that (T2) all trajectories in $\mathcal{X}$ are precompact, and (T3) the set $\mathcal{W}$ of ω-limit sets are countable. Those two assumptions may seem restrictive, as it excludes systems with diverging solutions or a continuum of equilibria. But even if $\mathcal{X}$ does not satisfy the assumptions, the results still hold as long as there exists a forward invariant subset of $\mathcal{X}$ that contains more than one ω-limit sets and satisfies those assumptions.

In addition, if a system does not have a one-to-one linear immersion on $\mathcal{X}$ containing multiple ω-limit sets, the examples in Section 3 suggest that a one-to-one linear immersion may still exist if we restrict the system to a subset of $\mathcal{X}$ that contains a single ω-limit set, or a domain of attraction in $\mathcal{X}$. Earlier works by Lan and Mezić (2013); Williams et al. (2015); Brunton et al. (2016); Bakker et al. (2019) suggest that one may want to learn a separate linear immersion for each domain of attraction of ω-limit sets in $\mathcal{X}$. This suggestion is supported by results in Lan and Mezić (2013), which proves that a continuous one-to-one linear immersion is guaranteed to exist in the domain of attraction of many different types of ω-limit sets, including stable or unstable equilibrium. Our results provide theoretical support for this suggestion from a different aspect since we show that in many cases the domain of attraction of one ω-limit set is the largest domain where a continuous one-to-one linear immersion exists.

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