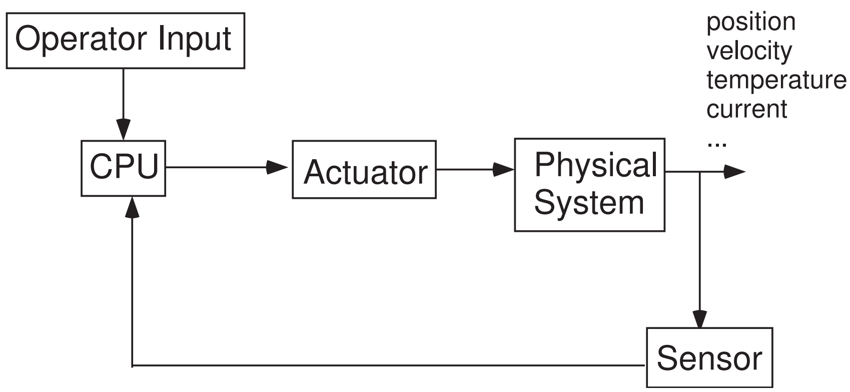
Feedback Systems

- Many embedded system applications involve the concept of *feedback*

- Sometimes feedback is *designed* into systems:



- Other systems have naturally occuring feedback, dictated by the physical principles that govern their operation

Feedback Systems

- Some examples we will see:
  - op-amp
  - motor equations: mechanical
  - motor equations: electrical
  - DC motor: back EMF
  - current controlled amplifier
  - velocity feedback control
- How many examples of feedback can you think of?

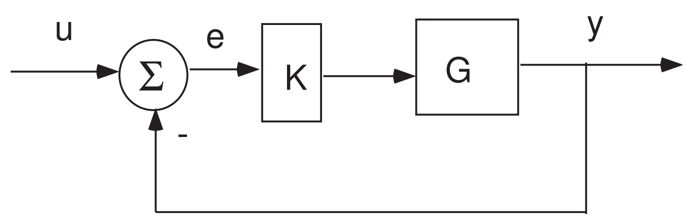
Issues with Feedback

- A feedback loop in a system raises many issues
  - requires a sensor!
  - changes gain
  - reduces effects of parameter uncertainty
  - may alter stability
  - changes both steady state as well as dynamic response
  - introduces phase lag
  - sensitive to computation/communication delay

- Detailed analysis (and design) of feedback systems is beyond the scope of our course, but we will need to understand these basic issues...

Feedback and Gain

- Using high gain in a feedback system can make output track input:



- feedback response:

$$y = \frac{KG}{1 + KG}u$$

- error response:

$$e = \frac{1}{1 + KG}u$$

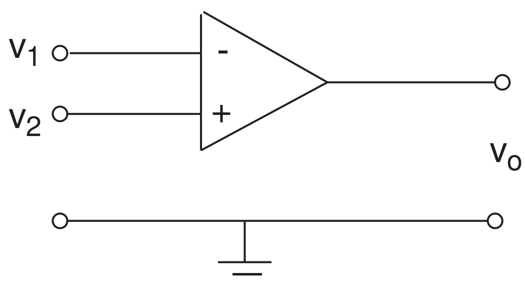
- high gain: as $K \to \infty$, $y \to u$ and $e \to 0$
  - "open loop gain": $|KG| \gg 1$
  - "closed loop gain": $|KG/(1 + KG)| \approx 1$

    $\Rightarrow$ we can make the output track the input *even if we don;t know the exact value of the open loop gain*!

- CAVEAT: only useful if system is stable!
  - for all but very simple systems, use of excessively high gain will tend to destabilize the system!
- a simple example where dynamics are usually ignored: op amp

Operational Amplifier (Op Amp)

- An op amp [2] is used in many electronics found in embedded systems. Hence it is of interest in its own right, as well as being a simple example of a feedback system



v_1, v_2: input voltages

v_o: output voltage

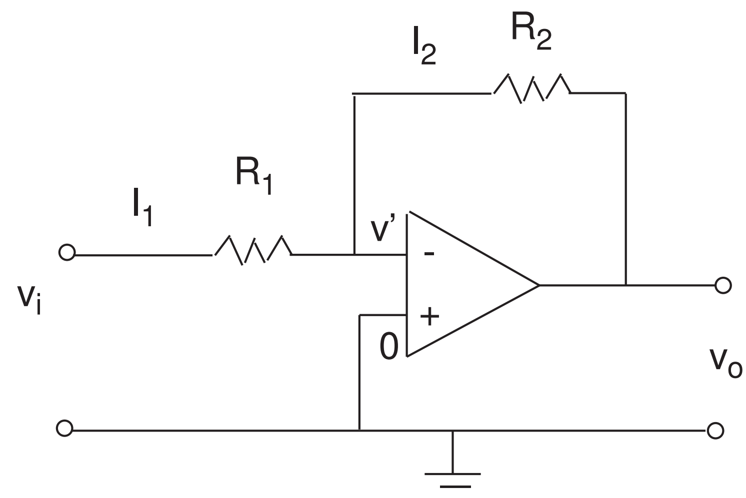
- output voltage depends on *difference* of input voltages

$$v_o = K(v_2 - v_1) = -K(v_1 - v_2)$$

- Typically $K \approx 10^5 - 10^6$, but varies significantly due to manufacturing tolerances
- Ideal op amp
  - no current flows into input terminals
  - output voltage unaffected by load
- In reality
  - op amp is a low pass filter with very high bandwidth
  - draws a little current
  - is slightly affected by load
- we shall assume an ideal op amp

Inverting Amplifier, I

- Q: How to use the op amp as an amplifier given that gain is uncertain?
- A: Feedback!

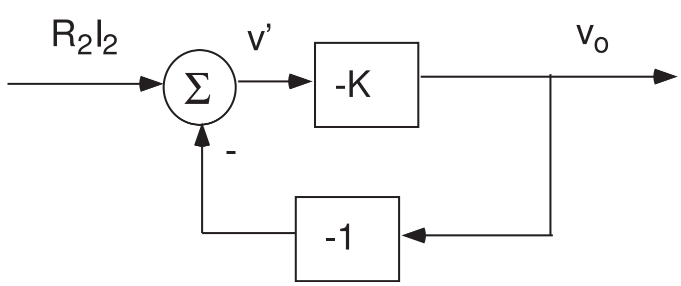


- currents:

$$I_1 = \frac{v_i - v'}{R_1}, \qquad I_2 = \frac{v' - v_o}{R_2}$$

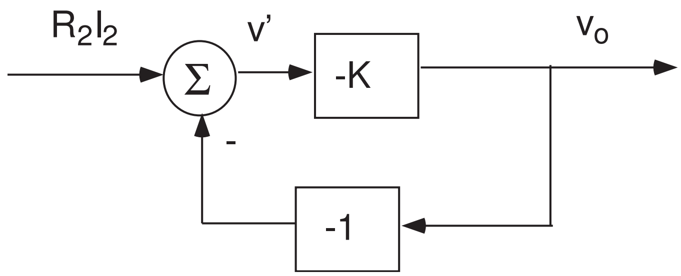
- feedback equations:
  - from previous page, v_o depends on v': $v_o = -Kv'$
  - v' depends on v_o: $v' = v_o + R_2 I_2$

  $\Rightarrow$



Inverting Amplifier, II

- Feedback diagram:



- Apply rule for transfer function of feedback system:

$$v_o = -\left(\frac{K}{1+K}\right) R_2 I_2$$

- If $K >> 1$, then the feedback equations imply that

$$v_o \approx -R_2 I_2$$

- It further follows that $v' = v_o + R_2 I_2 \approx 0$. By assumption that the op amp draws no current, $I_1 = I_2$, and thus

$$v_o = -\left(\frac{R_2}{R_1}\right) v_i$$

$\Rightarrow$ Feedback allows us to use an op amp to construct an amplifier without knowing the precise value of K!

More Complex Feedback Examples

- to analyze op amp, we ignored dynamics and treated the op amp as a pure gain that was constant with frequency
- in general, dynamics cannot be ignored
  - transient response
  - stability
- Two examples where feedback arises from the physics
  - motor dynamics: mechanical
  - motor dynamics: electrical
- we shall discuss these examples, but we will first consider a simple case: feedback around an integrator

Integrator

- Equations of integrator

$$\dot{x} = u$$

$$x(t) = x(0) + \int_0^t u(\sigma) d\sigma$$

- Examples:
  - u is velocity, x is position
  - u is acceleration, v is velocity
  - voltage and current through inductor: $I = \frac{1}{L} \int V dt$
  - voltage and current through capacitor: $V = \frac{1}{C} \int I dt$

- Integrator is an *unstable* system
  - the *bounded* input, $u(t) = 1$, yields the *unbounded* output

$$x(t) = x(0) + t$$

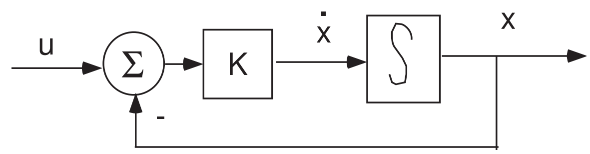
- Transfer function of an integrator

$$\int \Leftrightarrow \frac{1}{s}$$

$\Rightarrow$ integrator has infinite gain at DC, $s = 0$

Feedback Around an Integrator

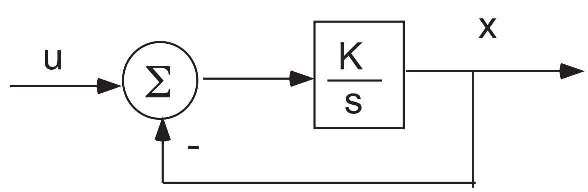
- Suppose there is feedback around integrator:



- differential equation of feedback system

$$\dot{x} = -Kx + Ku$$

- Transfer function of feedback system:



$$X(s) = \left(\frac{K/s}{1 + K/s}\right) U(s) = \left(\frac{K}{s + K}\right) U(s)$$

- The system is *stable* if $K > 0$.

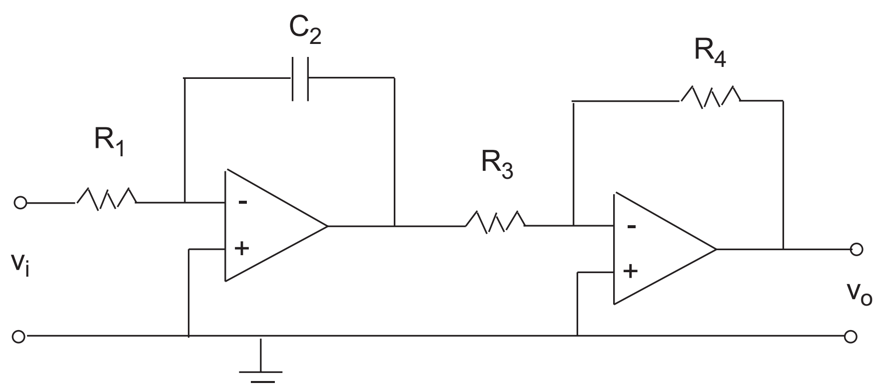
$\Rightarrow$ The response to the constant input $u(t) = 1$ yields

$$x(t) \to 1$$
$$\dot{x}(t) \to 0$$

independently of the value of K

Uses of an Integrator

- sometimes integrators arise from the physics
- other times they are constructed
  - to perform analog simulation of physical system
  - to add *integral control* to a system
- Op-amp integrator



  - Transfer function:

$$v_o = \frac{R_4}{R_3} \frac{1}{R_1 C_2 s} v_i$$

- Can also implement integrator on a microprocessor
  - discrete simulations
  - digital control

Motor Equations, Mechanical

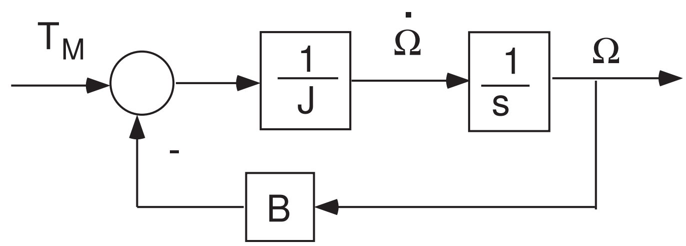
- equations of motion for shaft dynamics

$$J\dot{\Omega} = T_M - B\Omega$$

$$\dot{\Omega} = \left(\frac{1}{J}\right) T_M - \left(\frac{B}{J}\right)\Omega$$

Ω: shaft speed, $B \geq 0$: friction coefficient, $J > 0$: shaft inertia, T_M: motor torque

- Feedback diagram



- Transfer function:

$$\Omega(s) = \frac{\frac{1}{sJ}}{1 + \frac{B}{sJ}} T_M(s) = \frac{1/B}{sJ/B + 1} T_M(s)$$

- Constant torque $\Rightarrow$ speed goes to a steady state value:

$$\Omega_{ss} = T_M / B$$

- NOTE: with no friction ($B = 0$), system is unstable!
  - constant torque implies $\Omega(t) \to \infty$

Motor Equations, Electrical

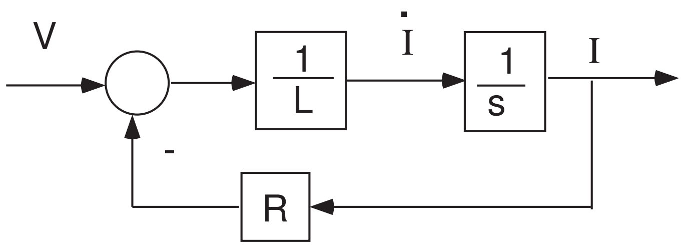
- equations of armature winding (ignoring back emf)

$$L\dot{I} = V - RI$$

$$\dot{I} = \left(\frac{1}{L}\right) V - \left(\frac{R}{L}\right) I$$

I: current, R: resistance, J: inductance, V: applied voltage

- Feedback diagram



- Transfer function:

$$I(s) = \frac{\frac{1}{sL}}{1 + \frac{R}{sL}} V(s) = \frac{1/R}{sL/R + 1} V(s)$$

- Constant voltage $\Rightarrow$ current goes to a steady state value:

$$I_{ss} = V/R$$

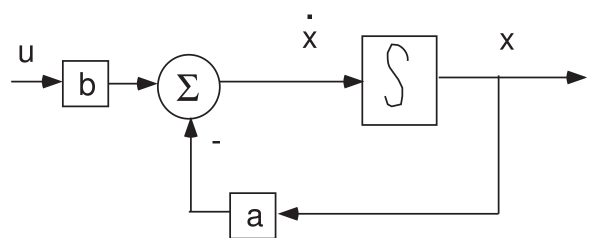
First Order Systems

- Shaft dynamics and circuit dynamics are each examples of a *first order systems*; i.e., they each have one integrator
- In general, a first order system may be written in the form

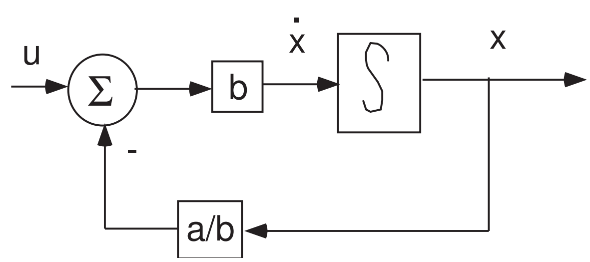
$$\dot{x} = -ax + bu$$

where x is the "integrator state", u is the input, and a and b are constants.

- Feedback diagram:



- Equivalently



- Transfer function:

$$X(s) = H(s)U(s)$$

$$H(s) = \left(\frac{b}{a}\right)\left(\frac{1}{s/a+1}\right)$$

Stability and Time Constant

- Time response:

$$x(t) = e^{-at}x(0) + \int_0^t e^{-a(t-\sigma)}bu(\sigma)d\sigma$$

- Response to a unit step, $u(t) = 1, t \geq 0$:

$$x(t) = \frac{b}{a}\left(1 - e^{-at}\right)$$

- The system is *stable* if $a > 0$
  - stability implies that $x(t) \to \frac{b}{a}$ as $t \to \infty$

- Rate of convergence determined by *time constant*, $\tau = 1/a$
  - at $t = \tau$, step response achieves 63% of its final value
  - at $t = 2\tau$, step response achieves 87% of its final value
  - at $t = 3\tau$, step response achieves 95% of its final value

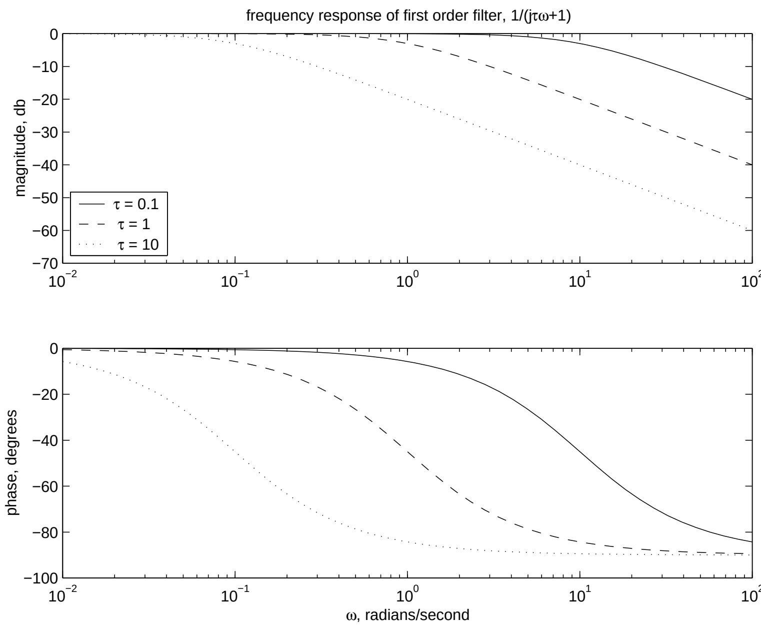
- To easily compare rate of convergence, normalize so that $b = a$
- Normalized frequency response:

$$x = H(j\omega)u, \qquad H(j\omega) = \left(\frac{1}{j\tau\omega + 1}\right)$$

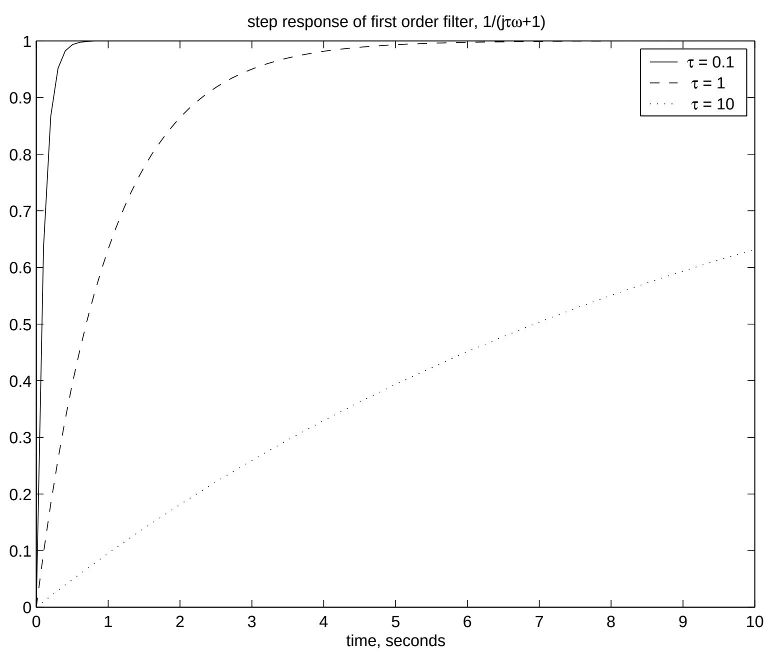
- NOTE: The time constant determines the rate at which the response of the system must be sampled in order to adequately represent it in digital form.

Bandwidth and Response Speed

- Time constant, τ determines[1]
  - bandwidth of frequency response:



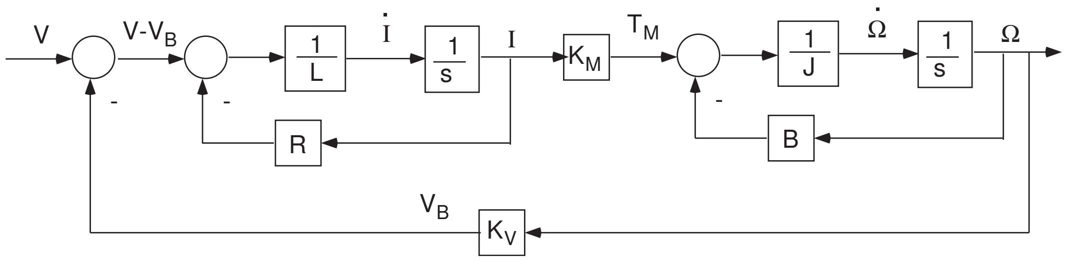
  - speed of response to unit step input, $u(t) = 1$:



[1] Plots created with Matlab file first_order.m.

Complete Motor Model

- The motor has both electrical and mechanical components, interconnected by the back EMF feedback loop:



- Two integrators $\Rightarrow$ a *second order* system

- Rules for combining transfer functions $\Rightarrow$

$$\Omega(s) = \frac{\left(\frac{1}{sL+R}\right)\left(\frac{K_M}{sJ+B}\right)}{1+\left(\frac{K_v}{sL+R}\right)\left(\frac{K_M}{sJ+B}\right)} V(s)$$

$$= \frac{K_M}{(sJ+B)(sL+R)+K_v K_M} V(s)$$

Second Order Systems

- Question: How to analyze and describe properties of second order systems?
  - stability
  - steady state response
  - transient response

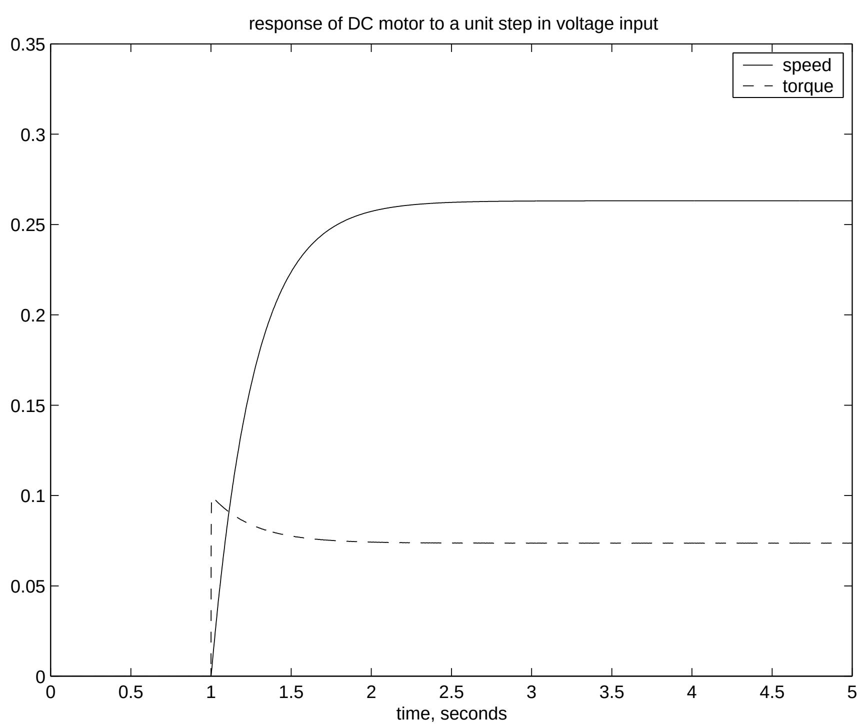
- Approach 1:
  - If the system can be decomposed into component first order subsystems, then (perhaps) properties of the overall system can be deduced from those of these subsystems.
  - Example: DC motor

- Approach 2: General analysis procedure.
  - Roots of characteristic equation
  - Damping coefficient and natural frequency determine response
  - Example: Virtual spring/mass/damper systems

$\Rightarrow$ We will need to understand the relation between transient response and characteristic roots (natural frequency and damping) in order to design force feedback algorithms in Lab 6!

Time Scale Separation

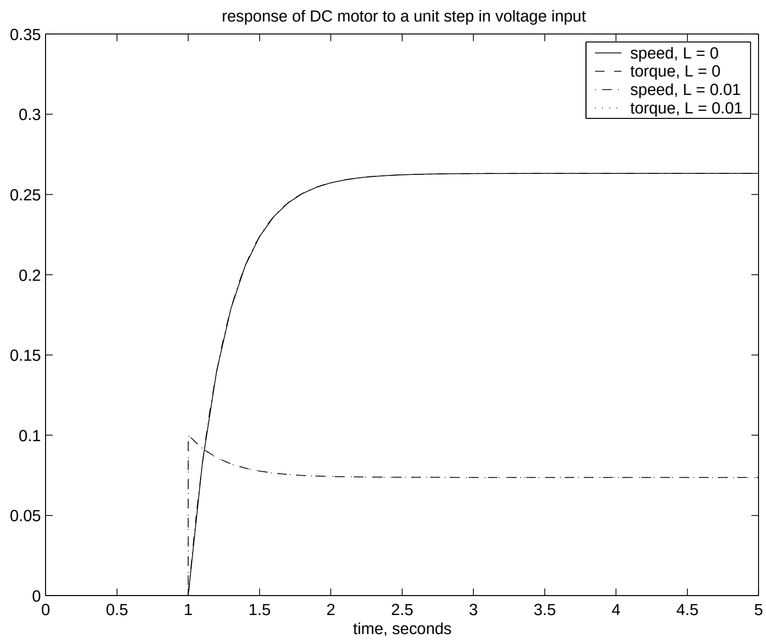
- For a DC motor, the time constants for each first order subsystem may be very different:
  - electrical subsystem: $\tau_e = L/R = 0.001$
  - mechanical subsystem: $\tau_m = J/B = 0.35$
- Mechanical subsystem is much slower than the electrical subsystem
  - Response of motor shaft is dominated by the mechanical subsystem
  - On the shaft speed time scale, current appears to be instantaneous
  - Since current and torque are related directly, $T_M = K_M I$, torque also responds rapidly[2]



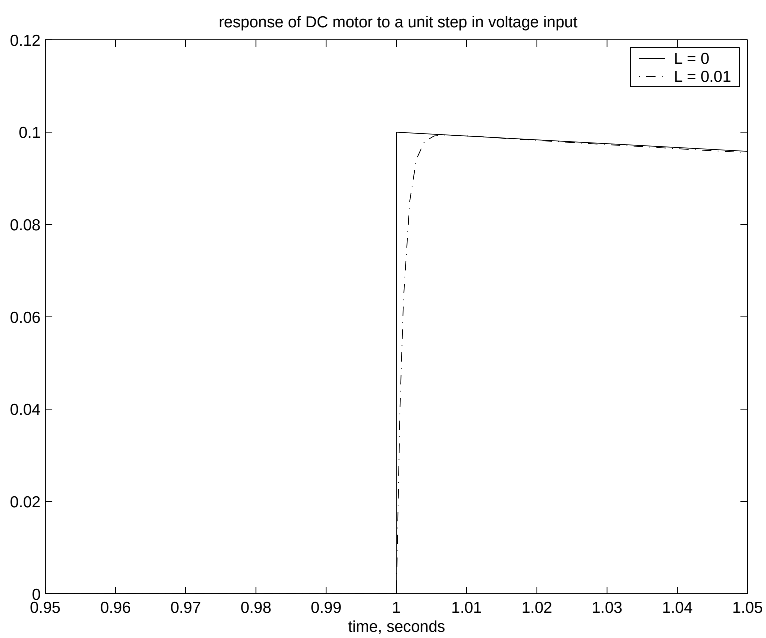
[2] Matlab files motor_linear.m and DC_motor_linear.mdl

Second Order Systems

- Electrical dynamics can be ignored by setting $L = 0$[3]



- Detail:



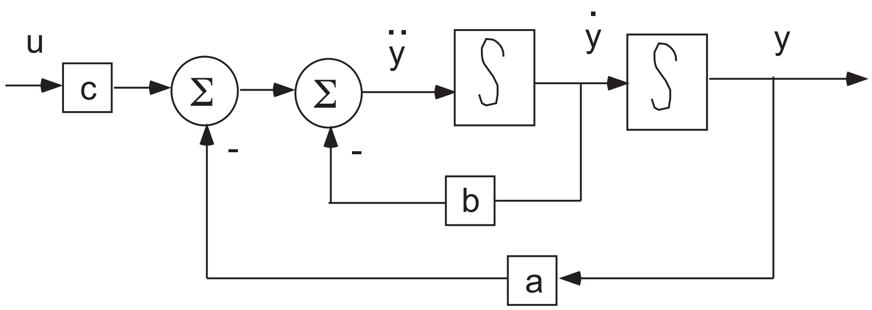
- Will need to model current when we implement torque control

[3] Matlab files motor_neglect_circuit.m and DC_motor_linear.mdl

Second Order Systems

- Systems with two integrators
  - DC motor
  - system with input and output described by the differential equation

$$\ddot{y} + b\dot{y} + ay = cu$$



- The frequency response function can be written as

$$H(s) = \frac{c}{s^2 + bs + a}$$

- Example: DC Motor

$$H(s) = \frac{\frac{K_M}{JL}}{s^2 + \left(\frac{BL+JR}{JL}\right) s + \left(\frac{BR+K_M K_V}{JL}\right)}$$

Characteristic Roots

- Suppose the frequency response is given by

$$H(s) = \frac{c}{s^2 + bs + a}$$

- Define the *characteristic equation*:

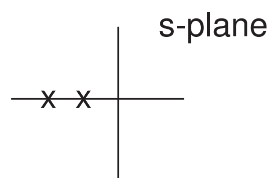
$$s^2 + bs + a = 0$$

- Characteristic roots

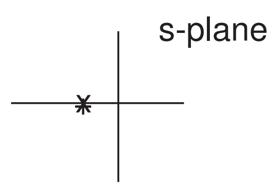
$$s = \frac{-b \pm \sqrt{b^2 - 4a}}{2} \tag{1}$$

- Possibilities:

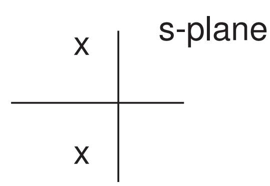
(i) $b^2 - 4a > 0 \Rightarrow$ two distinct real roots



(ii) $b^2 - 4a = 0 \Rightarrow$ one repeated real root

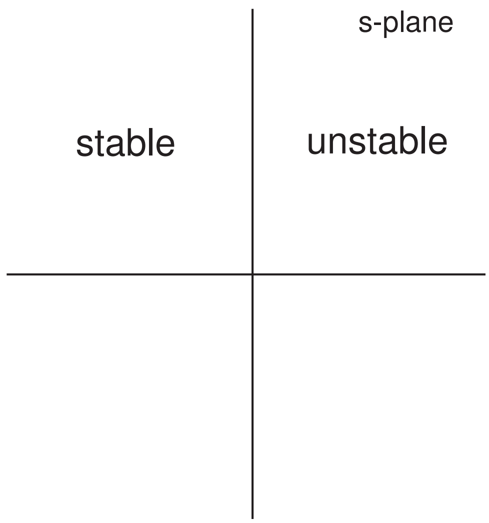


(iii) $b^2 - 4a < 0 \Rightarrow$ two complex conjugate roots



Characteristic Roots and Stability

- Second order system is
  - *stable* if the characteristic roots lie in the Open Left Half Plane (OLHP)
  - *unstable* if the characteristic roots lie in the Closed Right Half Plane (CRHP)
  - (roots on the imaginary axis are sometimes called *marginally stable*)



Natural Frequency and Damping

- Parameterize roots of $s^2 + bs + a = 0$ by

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{2}$$

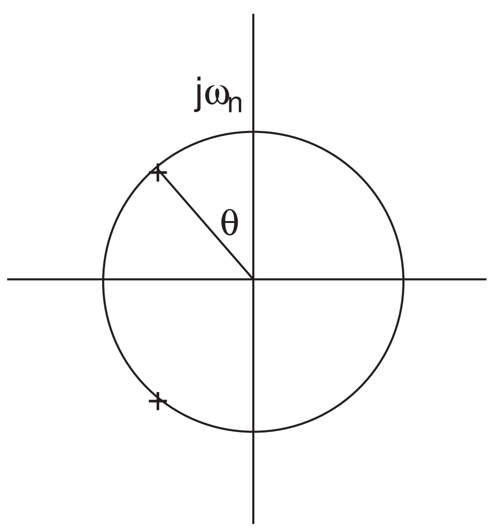
  where *natural frequency*, ω_n, and *damping coefficient*, ζ, are defined by (compare (2) with (1))

$$b = 2\zeta\omega_n, \qquad a = \omega_n^2$$

- roots lie on circle of radius ω_n at an angle

$$\theta = \arctan \zeta/\sqrt{1-\zeta^2}$$

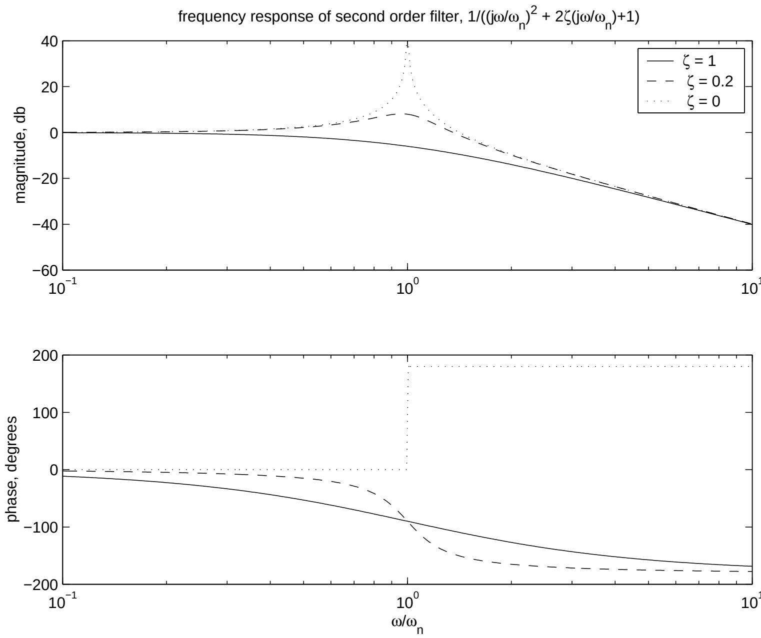
  with the imaginary axis:



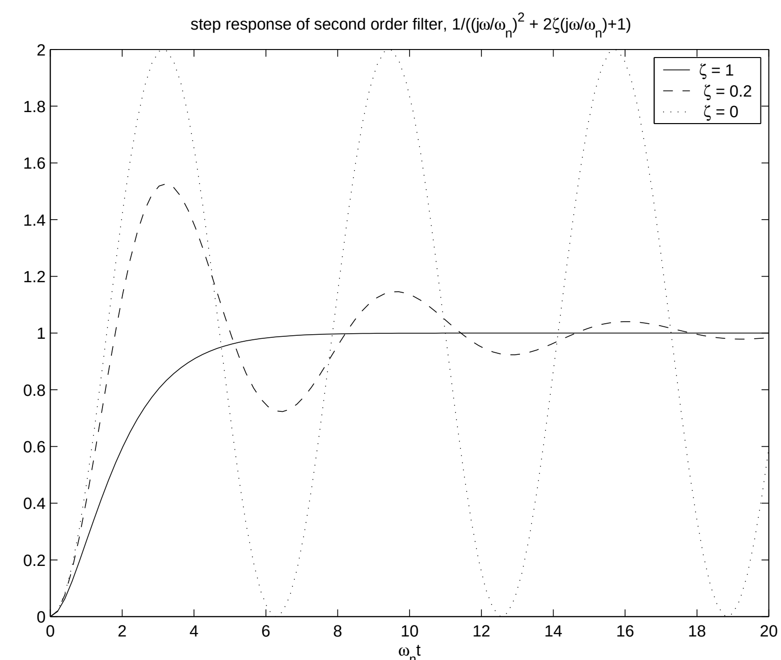
- Roots are
  - real if $\zeta^2 \geq 1$
  - complex and stable if $0 < \zeta < 1$
  - imaginary if $\zeta = 0$

Frequency and Time Response

- Natural frequency, ω_n and damping ratio, ζ determine[4]
  - bandwidth and peak of frequency response:



  - speed and overshoot of unit step response:



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[4]Plots created with Matlab m-file second_order.m.

General Systems

- The characteristic equation of an n-th order system will have n roots; these roots are either *real*, or they occur in *complex conjugate* pairs.
- The characteristic polynomial can be factored as

$$\prod_{i=1}^{N_R}(s+p_i)\prod_{i=1}^{N_C/2}(s^2+b_is+a_i)$$

- Each pair of complex roots may be written as

$$s_{i\pm}=\frac{-b_i}{2}\pm\frac{\sqrt{b_i^2-4a_i}}{2}=x_i\pm jy_i$$

and have natural frequency and damping defined from

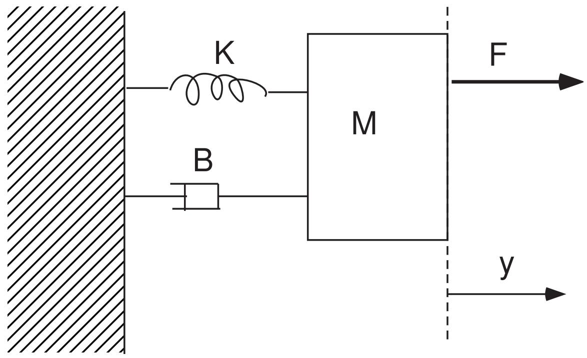
$$s_{i\pm}=-\zeta_i\omega_{ni}\pm j\omega_{ni}\sqrt{1-\zeta_i^2}$$

- Hence ζ and ω_n can be computed from the real and imaginary parts as

$$\omega_{ni}=\sqrt{x_i^2+y_i^2},\quad \zeta_i=-x_i/\omega_{ni}$$

- Note: It often happens that the response of a high order system is well approximated by one complex pair of characteristic roots.

Spring/Mass/Damper System

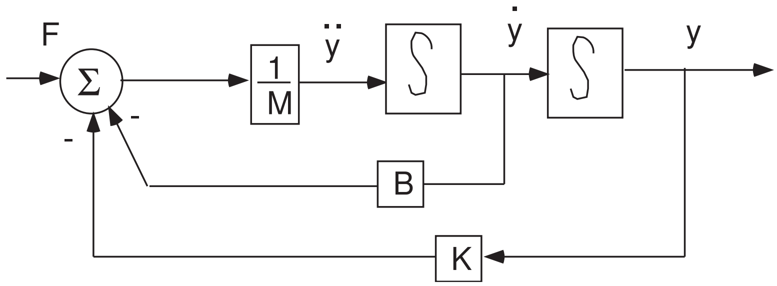


- Newton's laws:

$$M\ddot{y} + B\dot{y} + Ky = F$$

$$\Rightarrow \quad \ddot{y} = -\frac{B}{M}\dot{y} - \frac{K}{M}y + \frac{F}{M}$$

- Second Order System

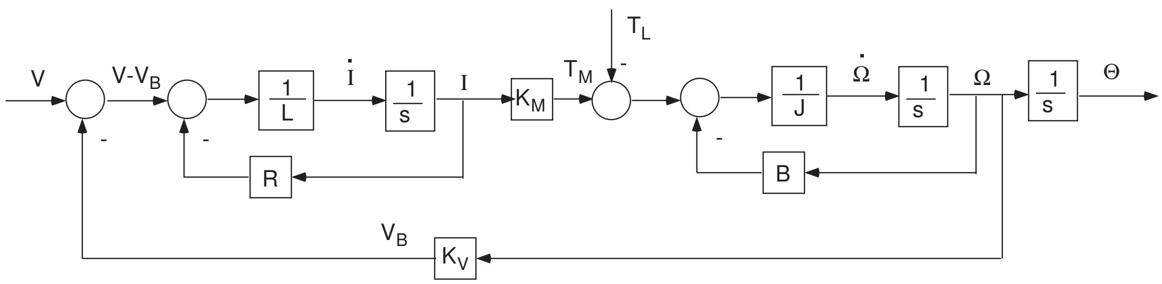


- Transfer Function:

$$Y(s) = \frac{\frac{1}{M}}{s^2 + \frac{B}{M}s + \frac{K}{M}}F(s)$$

Motor Control Strategies

- Can conceive of controlling four signals associated with the motor
  - input voltage, V
  - shaft position, Θ
  - shaft velocity, Ω
  - torque, T_M (equivalently, current, I)



- Issues:
  - Input (V) vs. output (Θ, Ω, I) variables
  - Open loop vs. feedback control (i.e., do we use sensors?)
  - Effect of load torque
  - Control algorithm (P, I, ...)
- Motor control results in higher order systems (more than two integrators)
- Higher order systems
  - Can still define characteristic polynomial and roots
  - Stability dictates that characteristic roots must lie in OLHP
  - Integral control may still be used to obtain zero error (provided that stability is present)
  - More complex control algorithms may be required to obtain stability

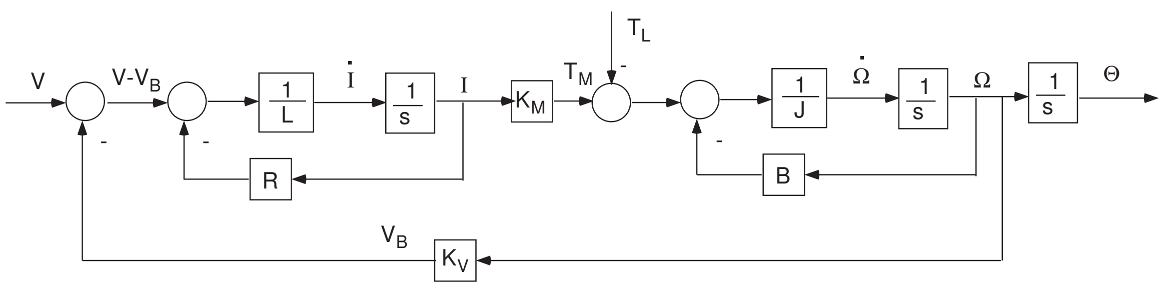
Voltage Control

- Apply desired V (either with a linear or a PWM amplifier)
- Suppose there is a constant load torque, T_L. Then steady state speed and torque depend on the load:

$$\Omega = \frac{K_M V - RT_L}{K_M K_V + RB}$$

$$T_M = \frac{K_M(VB + K_V T_L)}{K_M K_V + RB}$$

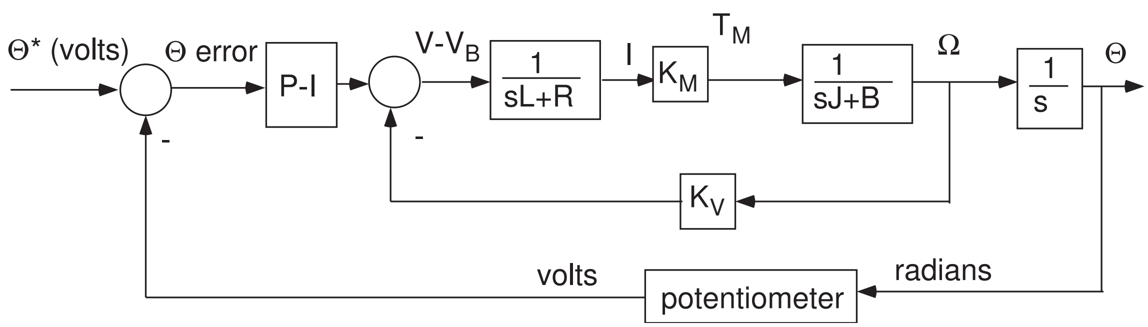
- Position $\rightarrow \infty$



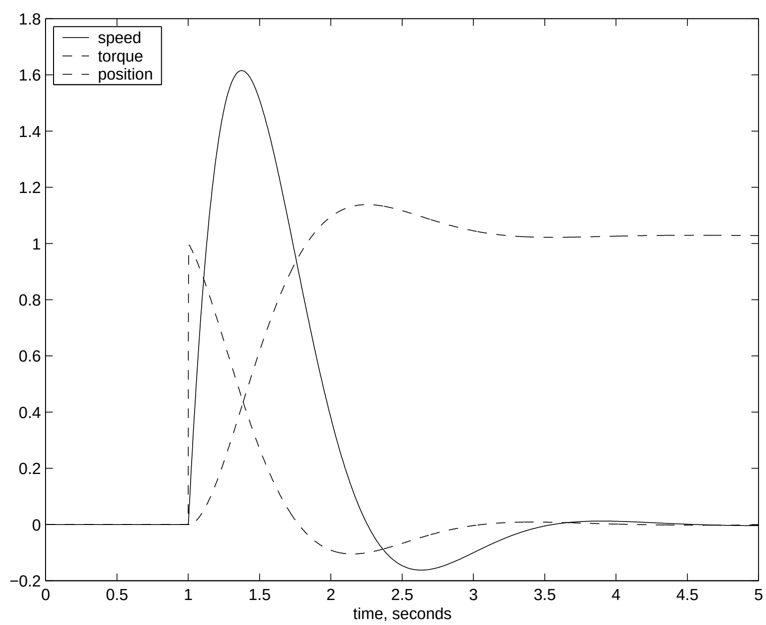
- Issues:
  - V is an input variable, and usually not as important as T_M, Θ, or Ω
  - Suppose we want to command a desired speed (or torque), independently of load or friction
    * Problem: usually load torque (and often friction) are unknown
  - Suppose we want to command a desired position
    * Problem: no control at all over position!

Position Control, I

- Suppose we want to control position
- We can use a sensor (e.g., potentiometer) to produce a voltage proportional to position, and compare that to a commanded position (also in volts).



- an integral controller cannot stabilize the system. Instead use a proportional-integral (P-I) controller: $10 + \frac{1}{s}$
- responses of speed, torque, and position due to a unit step command to position[5]



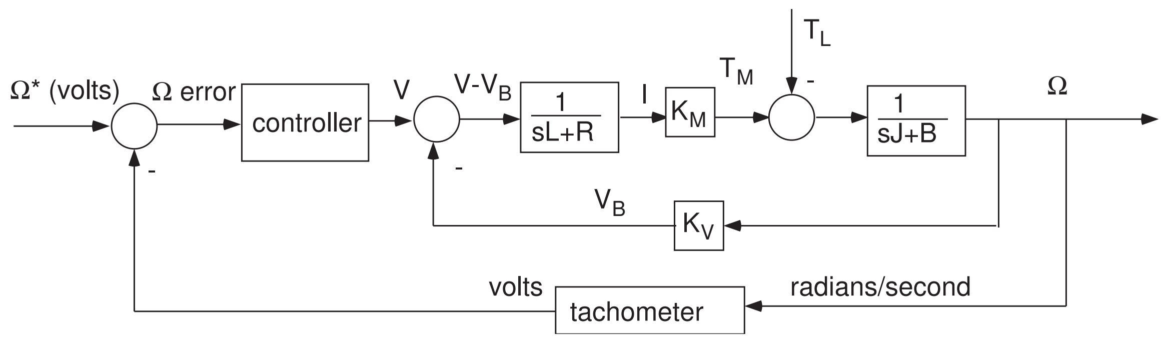
[5] Matlab files motor_position_FB.m and DC_motor_position.mdl

Position Control, II

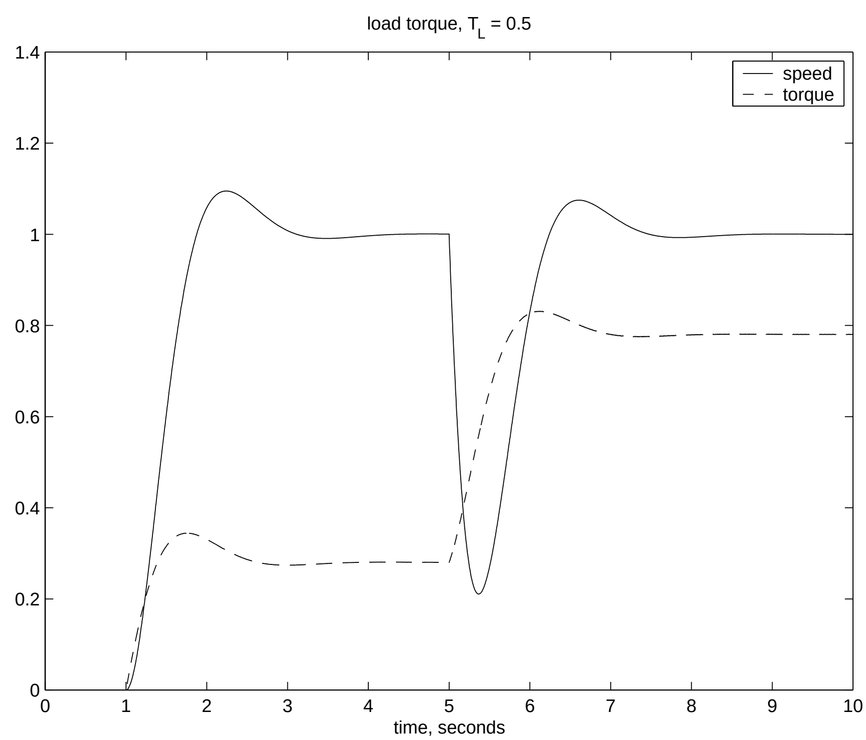
- P-I control: if feedback system is stable, then error approaches zero, and position tracks desired value
- Can implement analog P-I control using op amp circuit
- Control can also be implemented digitally using a microprocessor
- An encoder can be used instead of a potentiometer to obtain digital measurement
- PWM can be used instead of linear amplifier

Velocity Control, I

- Using an analog velocity measurement, from a tachometer, and an analog integral controller, allows us to track velocity



- despite the presence of an unknown load torque[6]

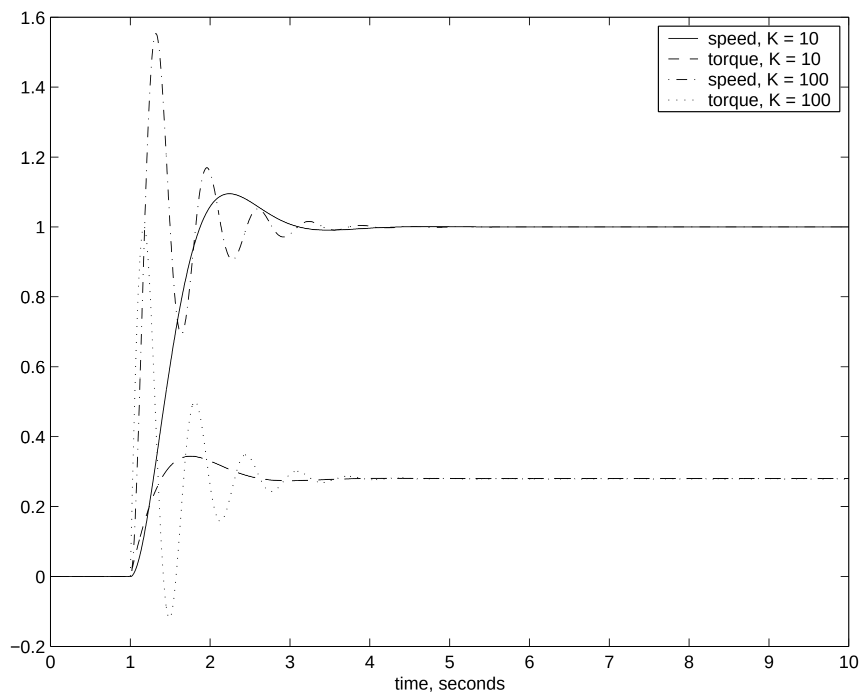


[6] Matlab files motor_speed_FB.m and DC_motor_speed.mdl

Velocity Control, II

- microprocessor control
  - use encoder measurement to generate digital velocity estimate
  - compare measured speed with desired speed
  - feed error signal into digital integral controller
  - generate PWM signal proportional to error

- Note: Performance depends on the controller gain[7]. Consider the difference between $10/s$ and $100/s$:

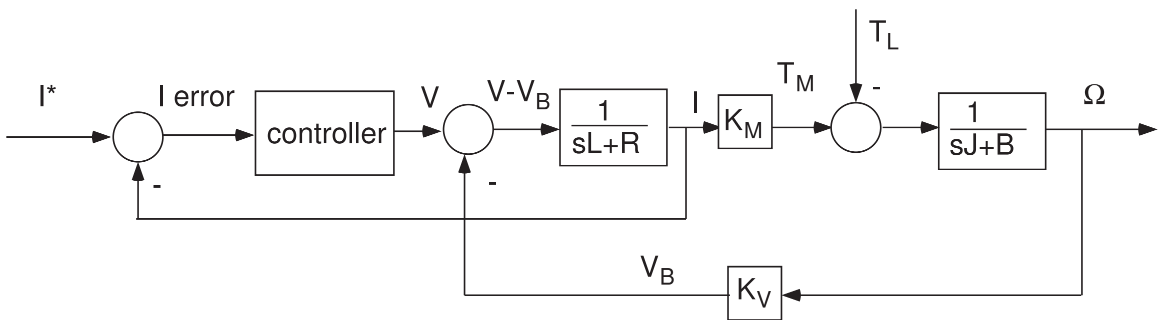


- Usually,excessively high gain leads to oscillatory response or instability!

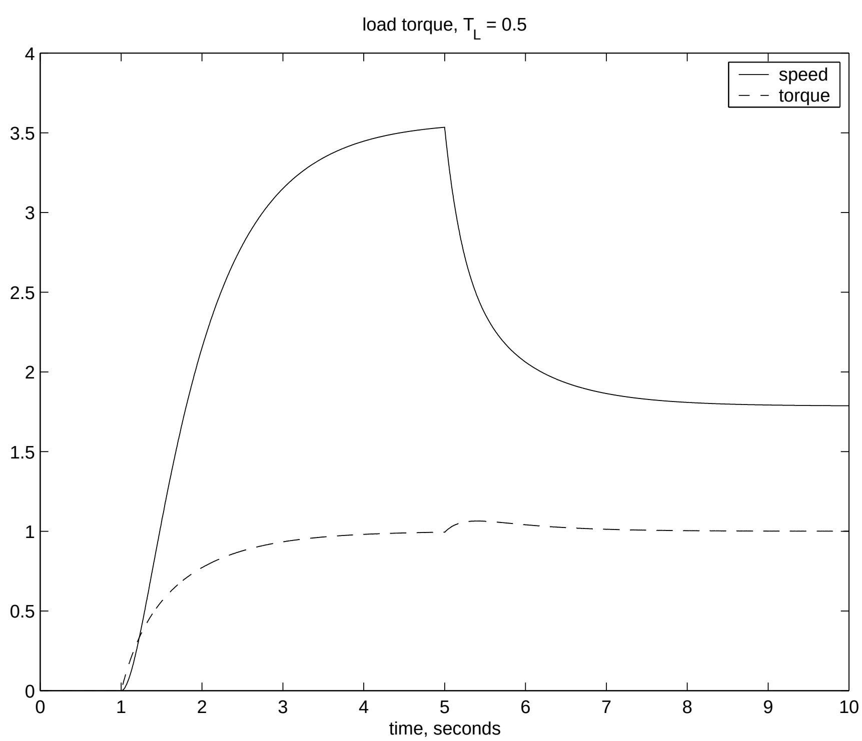
[7] Matlab files motor_speed_FB.m and DC_motor_speed.mdl

Torque Control

- Using a measurement of current and an analog integral controller, allows us to track torqe, which is directly proportional to current: $T_M = K_M I$



- despite the presence of an unknown load torque[8]

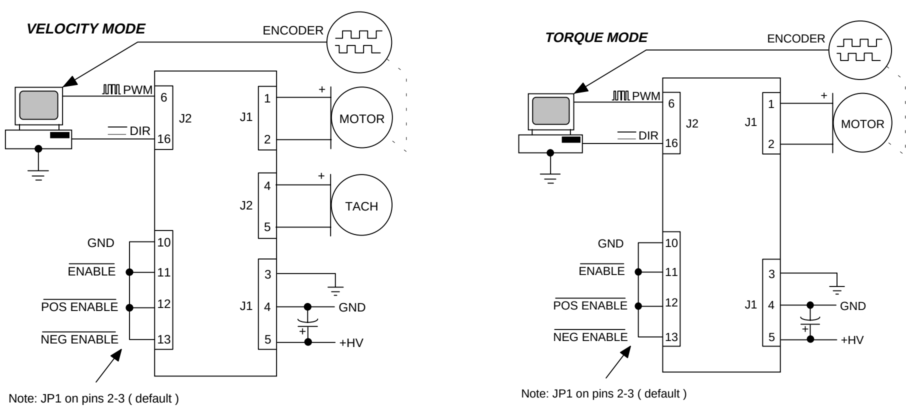


- Question: How does our lab setup implement torque control?

[8] Matlab files and motor_current_FB.m and DC_motor_current.mdl

PWM Amplifier, I

- Copley 4122D DC brush servo amplifier with PWM inputs [1]
- Two feedback control modes:
  - velocity control (requires a tachomoter)
  - torque (current) control
- We use torque control so that we can provide force feedback through our haptic interface

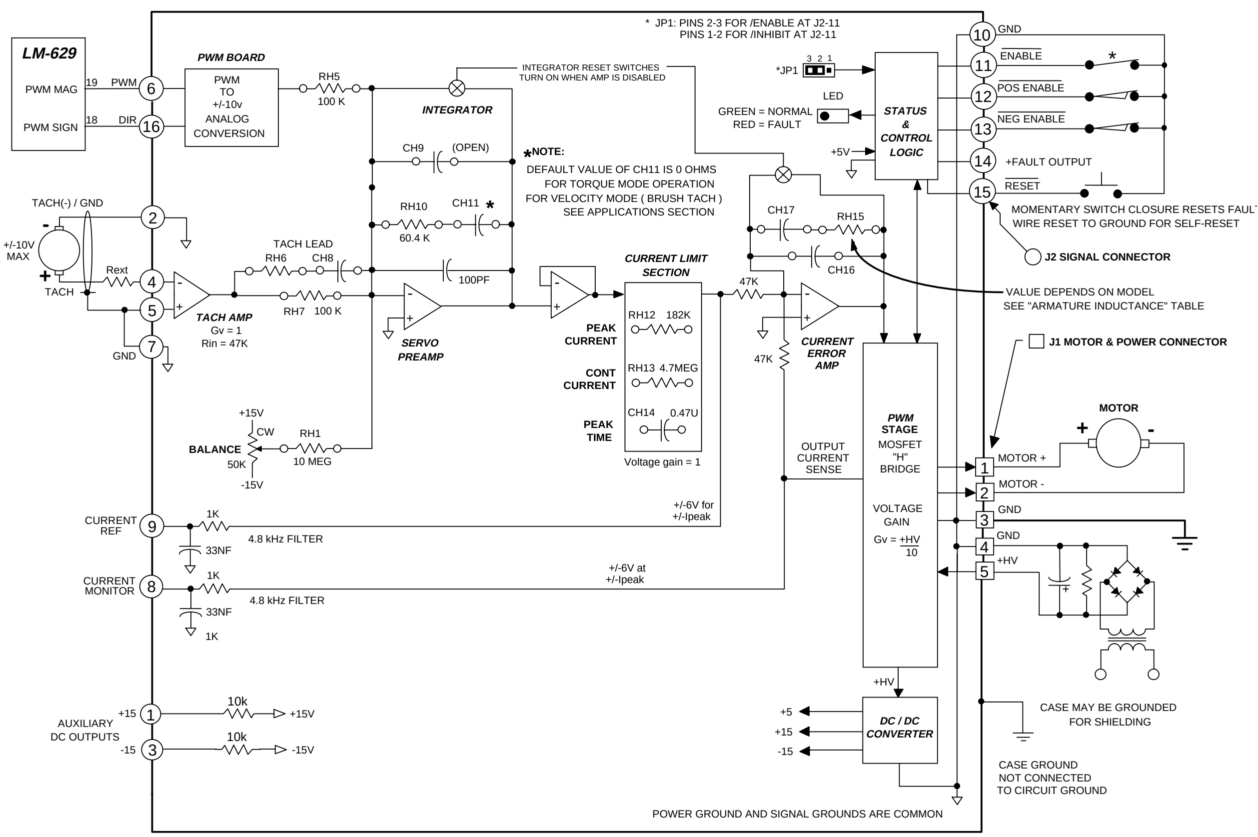


Notes

1. All amplifier grounds are common (J1-3, J1-4, J2-2, J2-7, and J2-10) Amplifier grounds are isolated from case & heatplate..
2. Jumper JP1 default position is on pins 2-3 for ground active /Enable input (J2-11)
   For /Inhibit function at J2-11 (+5V enables), move JP1 to pins 1-2
3. For best noise immunity, use twisted shielded pair cable for tachometer inputs.
   Twist motor and power cables and shield to reduce radiated electrical noise from pwm outputs.

PWM Amplifier, II

- “one-wire” mode: 50% duty cycle corresponds to zero requested torque
- analog integral controller with anti-windup
- H bridge PWM amplifier
- 25 kHz PWM output



References

[1] Copley Controls. Models 4122D, 4212D DC brush servo amplifiers with PWM inputs. www.copleycontrols.com.

[2] K. Ogata. *Modern Control Engineering*. Prentice-Hall, 3rd edition, 1997.