Convexity of the Set of Feasible Injections and Revenue Adequacy in FTR Markets

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***Abstract*—The feasible set of power injections for the constrained power flow equations is nonconvex when practical transmission capacity and bus voltage limits are imposed. The projection onto the space of active power injections may be "close" to convex, but this is not sufficient to guarantee revenue adequacy for the settlement of financial transmission rights.**

***Index Terms*—Convexity, financial transmission rights (FTRs), optimization methods, power flow analysis, power system economics.**

I. Introduction

THE ISSUE OF convexity of power flow solutions arises on occasion, particularly in relation to optimization problems such as economic dispatch [1], [2]. Sometimes convexity is discussed and used to justify results that lead to policy decisions, such as the establishment of "revenue adequacy" for congestion revenues in an electricity market [3], [4]. (Revenue adequacy ensures that the ISO will collect enough money in congestion rents to pay financial transmission rights.) Unfortunately, as we will constructively show, this set of power flow injections is typically nonconvex.

Analysis of the power flow equations is the most fundamental task in power system engineering. These equations represent, in a very quantitative sense, the physical capability of a network to transfer energy from injection points to end use (treated as negative injections). These equations apply to almost every type of problem that involves use of the electric power grid. In this paper, we are motivated by problems with roots in optimization in which a large set of possible injection profiles needs to be considered. This includes such problems as optimal dispatch to serve specified load at least cost and the operation of certain electricity-related financial markets such as the auction and settlement of financial transmission rights (FTRs).

In the well-known optimal dispatch problem, the injected powers at the loads are specified, as well as operational limits on line flow capacities, generator active and reactive power supply, and voltage magnitudes. The dispatch cost function that is to be minimized may or may not be convex. Detailed models of thermal generators tend not to have convex operating cost functions [5], and researchers continue to develop techniques for optimizing dispatch of these units [6]. However, in the electricity market setting in which suppliers submit strictly increasing offers, the costs functions are convex. If the set of feasible power injections is convex, efficient optimization algorithms can exploit these convex properties.

A related problem, and the main subject of this paper, is in the market of FTRs in which financial rights to congestion revenues for transferring quantities of active power from user-specified points of injection (POIs) to points of withdrawal (POWs) are auctioned. In an ideal setting, the auction maximizes income (from the auction) subject to a simultaneous feasibility test to ensure that the accepted FTRs satisfy the power flow equations. Since users can specify any POI and POW they choose, the entire set of feasible injections needs to be considered, at least in theory. There is the distinction in this case that the auction only considers active power, so we need to consider a restricted problem involving the projection of the feasible set of active powers, reactive powers, and voltage magnitudes and phase angles onto active powers, effectively ignoring convexity/nonconvexity in reactive power and voltage. Also, from a practical point of view, the auction should ideally mimic the actual system, so the POWs will generally correspond to load buses and the POIs to generator buses.

This issue of convexity has been discussed some in the literature. It is clear from the nonlinear form of the power flow equations in terms of voltage magnitudes and angles (polar coordinates) that the feasible set expressed in active power, reactive power, voltage magnitude, and angle cannot be convex. This is immediately obvious from the 2π-periodicity of the equations in terms of angles. Less obvious is whether the equations are nonconvex when voltages are expressed in rectangular coordinates, and moreover, we are primarily interested in power injections and do not necessarily require convexity in the voltage variables. In [7], Jarjis and Galiana conjectured that the set of active and reactive power injections is convex, based on unconstrained power flow equations expressed in rectangular coordinates. They make no claim to convexity when the network is constrained.[1] In his work on contract networks, Hogan recognized that the entire feasible set in power and voltages needs not be convex to prove revenue adequacy. Convexity in injected powers is sufficient and he conjectured that this set is convex [3, pp. 73–74]. Chao and Peck [4] proved that in the restricted case of constant voltages and small angle differences that the active power flows are convex functions of the angle differences and

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[1] Their work was directed toward finding the closest feasibility boundary to an operating point and did not consider the types of transmission capacity and voltage constraints that may be important for transmission congestion.

that the optimal dispatch problem is also convex under these conditions.

We comment that the general issue of conditions under which the set of active power injections allowed by the power flow equations is convex remains unresolved. Neither Jarjis and Galiana nor Hogan proved their conjectures, although they are based on appealing arguments. The conditions for which Chao and Peck establish convexity (constant nominal voltage, small angle differences) are generally not applicable when congestion management is important.[2] Furthermore, the range of angle differences for which their result holds (provably) depends on the ratio of line conductances to susceptances, reducing to zero for lossless systems.

In their overview of FTR mechanisms [8], Alsac *et al.* point out that revenue adequacy can be proved for the linear dc network model, but "no such proof is available for loss-compensated and/or nonlinear network models." In this paper, we show that revenue adequacy cannot be proven for the general nonlinear ac power flow model. The set of feasible power injections and voltage variables is not generally convex, nor is its projection onto the space of active power injections. Our paper builds on [9] and [10], where Hiskens and Davy used continuation methods to explore the boundary of the set of solutions to the power flow equations. Using a three-bus example, they showed that the set of feasible reactive power injections may contain holes and, hence, is nonconvex. It was also shown that line losses distort the set of feasible active power injections and are capable of introducing nonconvexity. We will explore these results further in this paper, showing nonconvexity in the set of feasible active power injections for the lossless case. Furthermore, by imposing transmission capacity and voltage constraints, we discuss the importance of nonconvexity in the context of transmission congestion and revenue adequacy.

In Section II, we present some mathematical background and provide an elementary example involving current injections into a linear network with line capacity and voltage limits. With the intuition gained from the current injection model, we proceed to examine the power injection model (power flow equations) in Section III. In both cases, we prove that the set of feasible injections is not convex. Section IV discusses the issue of convexity in terms of subsets that correspond to analyzes of practical interest. In Section V, we demonstrate that approximate convexity is not sufficient to support revenue adequacy. Conclusions are presented in Section VI.

II. Background

A. FTRs

When congestion occurs in a centrally dispatched, market-based system, the LMPs separate across the network, and consequently, in net, charges to loads for energy exceed payments to generators for supply. This congestion revenue in part or whole is intended to be disbursed to certain holders of FTRs.

FTRs are financial tools that allow market participants to manage their exposure to congestion costs. The details of FTRs depend on the market, but typically the right takes the form of a quantity of power flow between two points in the network: a POI and a POW.[3] The right is valued at market clearing as the product of the flow quantity and the price difference between the two points. Ideally, a load that is charged a premium for energy due to congestion can recoup its congestion charges by receiving payments from FTRs it may possess. The initial allocation of FTRs differs among ISOs. In the New York ISO, the FTRs are auctioned, and the proceeds of the auctions are distributed to the transmission owners. These funds count as part of the transmission owner's regulated profits and are thus, arguably, indirectly returned to the loads. In PJM, the loads have a right to request FTRs up to an amount equal to the load's peak demand. Any remaining FTRs, and FTRs the loads choose to sell, are distributed using an auction. The final FTR allocation is required to satisfy a "Simultaneous Feasibility Test" in which the specified power injections at POIs and POWs are feasible over the physical electrical network. (See [12, p. 78] for an example.)

An important issue for the ISO, which collects congestion revenues and pays out FTRs, is whether the congestion revenues will be sufficient to cover the FTR allocations. This is called revenue adequacy. An important theoretical result suggests that an auction-based FTR allocation that maximizes the income from the auction (and subject to a simultaneous feasibility test) will satisfy revenue adequacy provided the feasible set of power injections is convex [3]. Convexity is assured for the so-called dc power flow network representation, but it must be assumed to be true for the more accurate ac power flow network representation.

B. Convexity

Convexity is a mathematical property of a set that states that if one constructs a line between any two points in the set, all the points on the line will also belong to the set. The word *convexity* also describes a property of certain functions. A function $g(x)$ is said to be convex if for all x_1 and x_2 contained in a convex set, $g((1-\mu)x_1+\mu x_2) \leq (1-\mu)g(x_1)+\mu g(x_2)$ (see [13, pp. 82–83]). These important properties have been exploited in different ways to establish other properties (such as "revenue adequacy" mentioned above) and to develop efficient optimization routines.

Consider an optimization problem cast in the following way:

$$\min_P C(P) \tag{1}$$

subject to

$$f(P) \leq 0 \tag{2}$$

where P is a vector of variables, $C(P)$ is a scalar cost function expressed in terms of P, and $f(P)$ is a vector of constraints imposed on P that limits the values P may take. If $C(P)$ is a convex function and the feasible set $\Omega = \{P : f(P) \leq 0\}$ is also convex, then efficient algorithms exist to find an optimal solution, and the solution is guaranteed to be either unique or to

[2]When congestion occurs due to line or voltage constraints, the power flows and angle differences approach their allowable maxima, and/or the voltages deviate from nominal values by the maximum tolerable amount.

[3]In this paper, we consider point-to-point obligation FTRs as they are most commonly used. ERCOT employs a different flowgate FTR method, and other hybrid option/obligation, point-to-point/flowgate methods have been proposed but are not currently used. See [11] for an example.

belong to a continuous set of adjacent (feasible) minimal cost solutions. If either the cost function or the feasible set is not convex, then practical algorithms are not generally available to find the globally optimal solution. Only locally optimal solutions can be guaranteed. (It is shown in [14] that misapplication of sophisticated algorithms such as Lagrangian relaxation can result in suboptimal or infeasible answers, when the problem should exhibit a unique globally optimal answer.)

In this paper, we focus on the feasible set. In practice, it is implicitly defined by the more general constraints

$$f(P,Q,V) \leq 0 \tag{3}$$

where P, Q, and V, respectively, represent active power, reactive power, and voltage phasor. The voltage phasor may be represented in polar or rectangular coordinates. Given the nonlinear form of equality and inequality constraints contained in (3), it should not be expected that this optimization problem should be convex [15]. Since many typical and practical optimization problems focus only on active power, we can theoretically examine the projection of the set described by (3) onto active powers P to obtain (2). This resulting feasible set is a best-case representation since it may appear convex while the underlying representation with more variables may not be. If the feasible set described by (3) is not convex, then optimization algorithms may exhibit computational problems, as mentioned earlier. Nevertheless, if the projected feasible set described by (2) is convex, regardless of (3), one can establish useful theoretical properties. In this paper, we pursue the "revenue adequacy" result, which ensures that congestion revenues will be greater than or equal to FTR payments, an important property for the settlement of electricity markets. In [3], Hogan makes the point that the entire feasibility set in terms of power and voltages need not be convex to ensure revenue adequacy; convexity in injected power is sufficient.

C. Current Injection Example

Before we analyze the power flow problem, it is illustrative to first consider the current-injection network model from which the power flow model is ultimately derived.

In a simple form, neglecting complexities, including phase-shifting transformers, LTCs, and the like, the injected currents are related to bus voltages through a complex admittance matrix

$$\begin{bmatrix} I_1 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & \dots & Y_{1N} \\ \vdots & \ddots & \vdots \\ Y_{N1} & \dots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix} \tag{4}$$

or compactly $I = YV$, where I and V represent current and voltage complex phasors, respectively.[4] Kirchhoff's current law, a physical constraint that all the currents injected into the network must sum to zero, is implicitly included in this equation. If the network does not include any shunt elements, the admittance matrix will be singular. It is worth noting a physical interpretation for this singularity. The zero eigenvector of all ones indicates that any profile of identical voltages will result in zero current injections.

The set of feasible currents that satisfy (4) is convex (and infinite). This is easily confirmed by examining the current injections that form a line between two feasible solutions I_A and I_B

$$\begin{aligned} I_A &= YV_A \\ I_B &= YV_B \\ I(\mu) &= (1-\mu)YV_A + \mu YV_B = YV(\mu), \quad 0 \leq \mu \leq 1 \end{aligned}$$

where $V(\mu) = (1-\mu)V_A + \mu V_B$ explicitly describes the voltages related to these injected currents. No constraints are violated as μ varies between 0 and 1, so all points $I(\mu)$ are feasible. Of course, since this is a linear relation, we expect it to be convex. When we add certain constraints, this will not be true.

First, if we limit the magnitude of injected currents $|I_i| < I_{\max}$, the set of feasible injections remains convex. To see this, consider any two feasible current injections $|I_A|, |I_B| < I_{\max}$, and $I(\mu) = (1-\mu)I_A + \mu I_B$. Then because $0 \leq \mu \leq 1$

$$\begin{aligned} |I(\mu)| &\leq (1-\mu)|I_A| + \mu|I_B| \text{ norm property} \\ &< (1-\mu)I_{\max} + \mu I_{\max} = I_{\max} \end{aligned}$$

implying convexity.

Second, we may impose capacity limits on lines in the network. The relation between the line current flows and the bus voltages is similar in form to (4)

$$I_{\text{line}} = Y_{\text{line}} V \tag{5}$$

and following identical logic, the set of current injections that satisfy both maximum injection limits and maximum line capacity limits is convex.

Based on these observations, and recognizing that the power flow equations are simply a weighted version of the current equations (weighted by voltages, and the voltage magnitudes are nearly constant), intuitively it would seem that the set of feasible injections that satisfy the power flow equations should be convex. But the placement of constraints on the voltages undermines this—even in the linear current injection formulation.

Voltage constraints are only meaningful when the admittance matrix Y is nonsingular. As discussed earlier, singular Y implies that the voltage profile has translational symmetry along the zero eigenvector. In other words, the voltage profile can be arbitrarily shifted by adding the scaled zero eigenvector. Minimum voltage constraints can always be satisfied by raising the whole profile. Therefore, nonsingular Y will be assumed in this discussion. This assumption is consistent with realistic power systems, where π-models of transmission lines introduce shunt capacitance.

By placing lower and upper limits on voltage magnitude, it is easy to demonstrate that the set of feasible current injections cannot be convex. Consider two nonzero feasible solutions I_A and $I_B = -I_A$. The line $I(\mu) = (1-\mu)I_A + \mu I_B$ connecting these two injections will pass through zero at $\mu = 0.5$. It follows

[4]A tedious but important point: The complex current phasor variables are treated in rectangular coordinates and not polar coordinates. While this is not particularly meaningful with respect to current injections, when we extend the example to power injections, we will naturally need to use rectangular coordinates to distinguish between active and reactive power.

from (4) and nonsingularity of Y that $V = 0$ at this center point. This violates any nonzero minimum voltage magnitude limit we impose. Thus, with minimum voltage constraints, the set of injected currents that satisfy the network equations is decidedly nonconvex. The linear model represents a convex function of voltages, but the set of voltages when thus constrained is not convex. This leads to nonconvexity of the corresponding set of current injections.

Now, if lessons are to be learned from the linear current flow model, one might expect that the set of feasible power flow injections will suffer similar limitations. This is indeed the case and can be shown through a variant of the argument used above.

III. Nonconvex Set of Injected Powers

Let us turn to the power flow equations. These are constructed directly from the current injection equations by multiplying currents and voltages to obtain power

$$\begin{bmatrix} P_1 + jQ_1 \\ \vdots \\ P_N + jQ_N \end{bmatrix} = \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_N \end{bmatrix} \begin{bmatrix} Y_{11}^* & \dots & Y_{1N}^* \\ \vdots & \ddots & \vdots \\ Y_{N1}^* & \dots & Y_{NN}^* \end{bmatrix} \times \begin{bmatrix} V_1^* \\ \vdots \\ V_N^* \end{bmatrix} \tag{6}$$

or for each bus

$$P_i + jQ_i = V_i \sum_{k=1}^{N} Y_{ik}^* V_k^*. \tag{7}$$

To demonstrate that the set of power injections that satisfy (6) is not convex when minimum and maximum voltage constraints are imposed, we consider an elementary two-bus system and show that the set of feasible injections is not convex. We argue that this is sufficient to demonstrate problems with convexity for general power systems, because we can choose feasible operating conditions on a general system that allow it to be reduced to an equivalent two-bus system.

To this end, consider the two-bus system shown in Fig. 1. The two buses are connected through a lossless transmission line with reactance X. We neglect losses, but the reader will observe that the fundamental results that follow do not change with the addition of losses. The relevant power flow equations for this system are

$$P_1 + jQ_1 = \frac{jV_1}{X}\left(V_1^* - V_2^*\right) \tag{8}$$

$$P_2 + jQ_2 = \frac{jV_2}{X}\left(V_2^* - V_1^*\right). \tag{9}$$

Now we need to specify two specific feasible solutions to consider. Analogous to the current injection example of the previous section in which we reversed the current flow, here we choose feasible operating points that reverse the active power flow while keeping the reactive power constant. Let us define V_m to be the greater of the two minimum voltage limits for the two buses. The two cases we propose here are as follows.

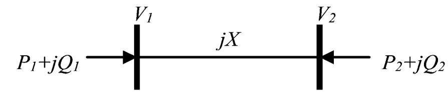

Fig. 1. Two-bus system.

Feasible Point A: $V_1 = V_m e^{j0} = V_m$ and $V_2 = V_m e^{j\pi/2} = jV_m$, giving

$$P_{A1} = -\frac{V_m^2}{X} \quad Q_{A1} = \frac{V_m^2}{X}$$
$$P_{A2} = \frac{V_m^2}{X} \quad Q_{A2} = \frac{V_m^2}{X}.$$

Feasible Point B: $V_1 = V_m e^{j0} = V_m$ and $V_2 = V_m e^{j3\pi/2} = -jV_m$, giving

$$P_{B1} = \frac{V_m^2}{X} \quad Q_{B1} = \frac{V_m^2}{X}$$
$$P_{B2} = -\frac{V_m^2}{X} \quad Q_{B2} = \frac{V_m^2}{X}.$$

From (8) and (9), candidate injections at bus 1 along the line connecting feasible points A and B are given by

$$\begin{aligned} P_1(\mu) + jQ_1(\mu) &= (1-\mu)P_{A1} + \mu P_{B1} \\ &\quad + j[(1-\mu)Q_{A1} + \mu Q_{B1}] \\ &= (1-\mu)\frac{V_m^2}{X}(-1+j) + \mu\frac{V_m^2}{X}(1+j) \\ &= \frac{jV_1(\mu)}{X}\left(V_1^*(\mu) - V_2^*(\mu)\right). \end{aligned} \tag{10}$$

Likewise for bus 2 injections, we obtain

$$\begin{aligned} P_2(\mu) + jQ_2(\mu) &= (1-\mu)P_{A2} + \mu P_{B2} \\ &\quad + j[(1-\mu)Q_{A2} + \mu Q_{B2}] \\ &= (1-\mu)\frac{V_m^2}{X}(1+j) + \mu\frac{V_m^2}{X}(-1+j) \\ &= \frac{jV_2(\mu)}{X}\left(V_2^*(\mu) - V_1^*(\mu)\right). \end{aligned} \tag{11}$$

Some algebra yields the necessary voltage profile along the path of injections

$$V_1(\mu) = V(\mu) \tag{12}$$
$$V_2(\mu) = V(\mu)e^{j\theta(\mu)} \tag{13}$$

where

$$V(\mu) = \frac{V_m}{\sqrt{2}}\sqrt{(2\mu-1)^2+1} \tag{14}$$

$$\theta(\mu) = -\arctan\left(\frac{2(2\mu-1)}{(2\mu-1)^2-1}\right). \tag{15}$$

Note that θ varies from $\pi/2$ to $3\pi/2$ by a path that passes through π at $\mu = 0.5$. The minimum voltage magnitude along this path occurs at $\mu = 0.5$

$$V_1(0.5) = \frac{V_m}{\sqrt{2}}, \quad V_2(0.5) = -\frac{V_m}{\sqrt{2}}. \tag{16}$$

Clearly at this point along the path of candidate injections, the minimum voltage constraint at one of the buses is violated. In fact, all of the injections along the path violate that minimum voltage constraint, except the endpoints. Therefore, the set of

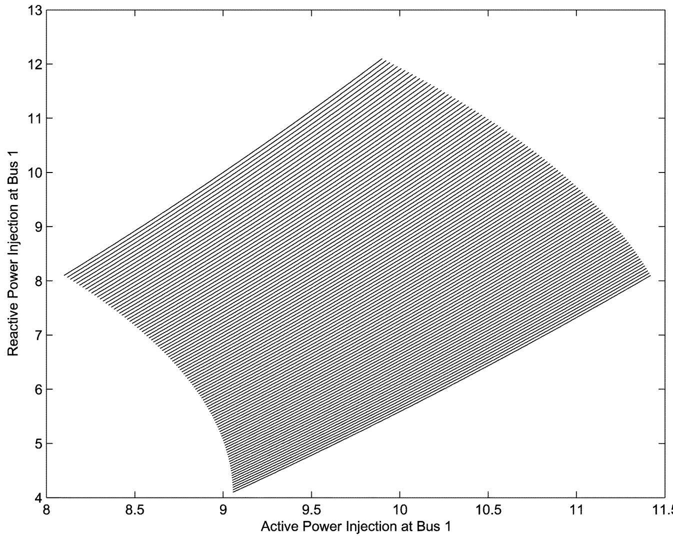

Fig. 2. Positive active and reactive power injections at bus 1 for the two-bus example.

feasible power injections for this two-bus system is not convex. More complex power systems can be thought of as composed of two-bus subsystems. Therefore, the nonconvex result applies to a very large class of power system models.

We should comment on the choice of the two initial feasible points chosen for this proof. One observes that bus 2 supplies active power to the network at Point A and absorbs active power from the network at Point B. Historically, two such operating points for the same system may not have been considered reasonable. These days, when generators are commonly sited at industrial plants, universities, and other distributed locations, it is indeed possible that the plant may be supplying power to the grid at some times and drawing power at other times. The two points chosen are not extreme. However, in practice, one may wish to limit the set of injections and enquire whether a particular subset is convex.

To conclude this section, we present a sample feasibility region for the two-bus example presented here. We stipulate minimum and maximum voltages limits to be 0.9 and 1.1 p.u., respectively, for both buses. The line impedance was set to 0.1 p.u., and the reactive power injection at bus 2 fixed at 8.1 p.u. A homotopy (continuation) method [9], [16] was used to determine the set of active and reactive power injections at bus 1 that satisfy the voltage constraints and the reactive power constraint at bus 2. This produced two disjoint sets: one with positive active power injections at bus 1 and a symmetric reflection of this with negative power injections at bus 1. Obviously, two disjoint sets cannot form a single convex set. Fig. 2 shows only the positive power injection portion. It is clear that even this half of the complete set is not convex.

IV. Subsets, Subspaces, and Projections

We know from our proof above that the entire set of feasible injections is not convex, but a subset might be. It is not difficult to specify constraints that will result in a convex set of injections—one can generally specify a small enough region around a feasible injection profile to construct a small feasible convex set. In [4], Chao and Peck show convexity in terms of

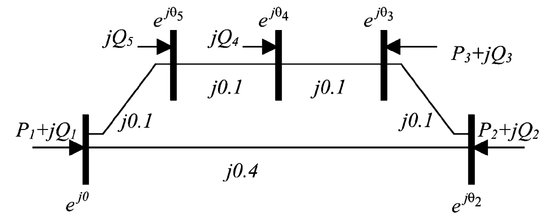

Fig. 3. Five-bus system with zero active power injections at buses 4 and 5 and all bus voltages equal to 1.0 p.u.

small angle differences and constant voltage. In [17], Sarić and Stanković suggest that the set of constant power factor injections in a range around a feasible operating point is convex. There is no point in trying to prove that a convex set will not exist for any possible constraint combination. Rather, we examine a particular case to consider some of the constraint issues that arise in practice. It should be alarming then to consider that the most elementary power flow problems that are encountered in an introductory power system course are nonconvex.

When we first learn or teach the power flow equations, we do not consider the entire space of possible injections. Rather we specify exactly enough binding constraints to ensure that the number of unknowns equals the number of equations. In an introductory power course, we learn to specify load bus constraints in terms of active and reactive power injections, generator buses in terms of injected active power and voltage magnitude, and we include a "slack bus" at which we specify an angle reference and voltage magnitude. Given these constraints, we attempt to find meaningful solutions in terms of reactive power injections and angles at generator buses, voltage magnitudes and angles at load buses, and active and reactive power at the slack bus. In terms of power injections, we consider a subspace of possible injections for the generator reactive powers and the slack bus active and reactive power. [Recall that load active and reactive powers are fixed, along with generator (except slack) active powers.] It is well known that the number of distinct solutions to this problem is finite [18]. Therefore, it is impossible for the set of feasible injections to be convex for this very constrained case (except for very rare cases when there is a unique solution). The (infinite) number of points comprising a line connecting two feasible injections cannot belong to a finite set.

But perhaps we should not be too concerned with the constraints imposed in the pedagogical model, because they do not reflect some of the most important uses in practice. That model does not include practical limits on voltages at load buses, and the generator powers are much more tightly constrained than in optimal dispatch and similar applications.

To establish a model that resembles the type encountered in practical optimization studies, and in particular the type that may be encountered in the allocation and receipt of congestion revenues, we specify some loads, all voltages, and examine the projection onto active powers. Thus, we consider a subset of feasible injections and examine the projection issue. The five-bus network shown in Fig. 3 will be used for illustration. It is motivated in part by the many three-bus systems studied in the power system economics literature, where loop flow is of interest [3], [4], [14], [19], [20, pp. 397–398]. Such effects cannot be replicated in two-bus models. The five-bus system incorporates the features required to study loop flows but adds additional buses that provide voltage support. We will show that when voltages are constrained, the feasible set of active power injections is not convex.[5]

In this system, the bus voltage magnitudes are all equal to 1.0 p.u. The voltage angle at bus 1 is arbitrarily set to 0.0 radians to provide the system angle reference. The active power injections at buses 4 and 5 are set to zero, and we investigate the entire set of possible active power injections at buses 1, 2, and 3.

The topology and transmission line reactances are chosen to appropriately represent a spatially distributed system with parallel paths (that allow "loop flows"). The line reactance of 0.4 p.u. connecting buses 1 and 2 is equal to the sum of the four 0.1 p.u. reactances along the other path that connects these buses. Evaluating the set of possible active power injections is easily accomplished by varying θ_2 and θ_3 through all possible combinations between 0 and 2π. For each value of θ_3, there are six possible values for θ_4 and θ_5 that satisfy the zero active power injection constraint at these two buses. To derive these values, note that the zero-power injection condition at bus 4 requires $\sin(\theta_4 - \theta_5) = \sin(\theta_3 - \theta_4)$ and the zero-power injection condition at bus 5 requires $\sin(\theta_5) = \sin(\theta_4 - \theta_5)$. Each of these can be expressed directly in terms of the angle variables. Accounting for angular modularity of 2π, each has two distinct representations. For bus 4 these are

$$(\theta_4 - \theta_5) = (\theta_3 - \theta_4) + k2\pi \text{ or} \tag{17}$$
$$(\theta_4 - \theta_5) = \pi - (\theta_3 - \theta_4) - k2\pi \tag{18}$$

while for bus 5 they are

$$(\theta_5) = (\theta_4 - \theta_5) + m2\pi \text{ or} \tag{19}$$
$$(\theta_5) = \pi - (\theta_4 - \theta_5) - m2\pi. \tag{20}$$

Solving for θ_4 and θ_5 in terms of θ_3 using the four possible combinations obtained from choosing one of either (17) or (18), and one of (19) or (20), one obtains the following six possible values:

$$\theta_4 = \frac{2\theta_3}{3} \text{ and } \theta_5 = \frac{\theta_3}{3}, \text{ or} \tag{21}$$
$$\theta_4 = \frac{2(\theta_3 + 2\pi)}{3} \text{ and } \theta_5 = \frac{(\theta_3 + 2\pi)}{3} \text{ or} \tag{22}$$
$$\theta_4 = \frac{2(\theta_3 - 2\pi)}{3} \text{ and } \theta_5 = \frac{(\theta_3 - 2\pi)}{3} \text{ or} \tag{23}$$
$$\theta_4 = \pi \text{ and } \theta_5 = -\theta_3 \text{ or} \tag{24}$$
$$\theta_4 = \pi \text{ and } \theta_5 = \theta_3 - \pi \text{ or} \tag{25}$$
$$\theta_4 = 2\theta_3 \text{ and } \theta_5 = \theta_3 - \pi \tag{26}$$

with (21)–(23) coming from the (17) and (19) pair.

These different angle combinations correspond to different possible reactive power injections that satisfy the zero active power injection requirement at buses 4 and 5. We are primarily concerned with the set of active power injections at buses 1, 2, and 3, and we initially consider all the angle possibilities (21)–(26).

[5]A three-bus system of the type that one finds in many papers on FTRs does not suffice to demonstrate nonconvexity. The set of active power injections appears convex, excluding losses. The entire set of injections, including reactive power, is not necessarily convex though. Reference [9] provides examples of nonconvexity in reactive power and active power when losses are considered. Our example here is lossless and ignores nonconvexity in reactive power and voltage.

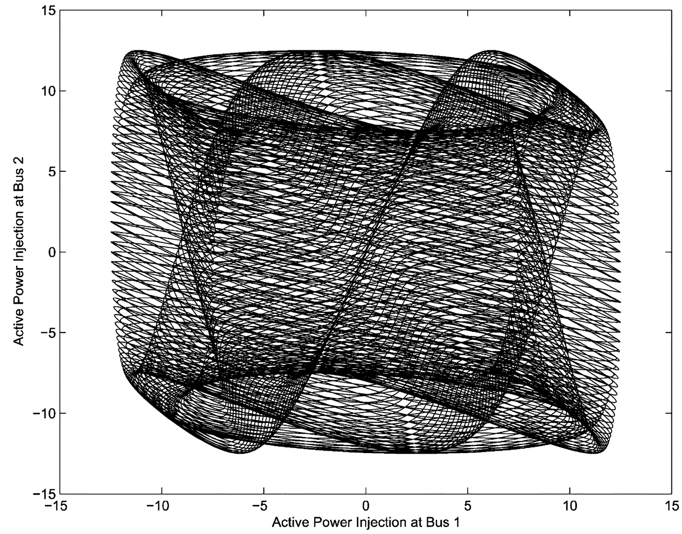

Fig. 4. Set of allowable active power injections at buses 1 and 2 for the system shown in Fig. 3.

The set of possible active power injections for buses 1 and 2 is shown in Fig. 4. Since there are no losses, the active power injection at bus 3 is the negative of the sum of active power injections at buses 1 and 2. In an FTR auction context, the set of injections shown in Fig. 4 can be interpreted as the possible point-to-point FTRs from buses 1 and 2 to bus 3 that satisfy a simultaneous feasibility test. This set is obviously nonconvex but appears "close" to convex. (We discuss the import of "close" to convex in Section V.) The manner in which the figure seems to wrap around is due to the different values of reactive power injections, which are not shown in this figure.

So far we have only imposed voltage constraints on the system. It may be appropriate to impose transmission line capacity constraints and power injection limits. Next we impose line capacity constraints that will allow maximum utilization of the lines to transmit active power and the necessary supporting reactive power but no active/reactive power combination beyond that: The angles between buses are restricted to be less than or equal to 90 degrees. We may also assume knowledge that buses 1 and 2 correspond to generator locations and bus 3 is a load location and look at the set when the active power injections at the generators are greater than or equal to zero. The set of feasible active power injections is shown in Fig. 5. Like the set shown in Fig. 4, this set is not convex (but may be considered "close" to convex). We continue this example in Section V where we compare congestion revenues with possible FTR obligations.

It might be argued that for almost all practical problems, the set will be close enough to being convex to justify algorithms and policies that assume a convex set. It should be understood, however, that the sets may not be completely convex, and problems may arise under some circumstances.

There is a notable study [21] concerning power transfer distribution factors (PTDFs) that suggests that the set of active powers will be nearly convex. The curvature seen on the boundary of the set of feasible injections in Fig. 5 is due in part to changes in the relative power transfer capabilities of the different lines in the network as the loading changes. Reference [21] shows

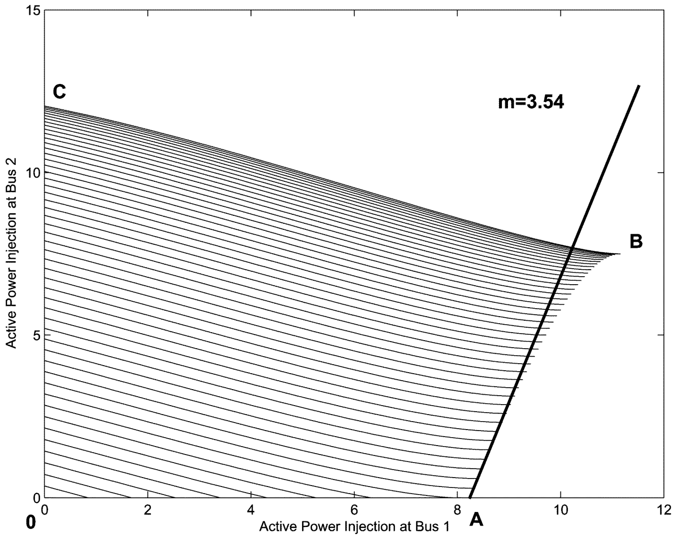

Fig. 5. Set of feasible active power injections at buses 1 and 2 with transmission line capacity limits.

that these relative power transfer capabilities (the PTDFs) are approximately constant, which is consistent with the feasibility region we find. Looking at Fig. 5, it can easily be seen that the boundaries are nearly linear, with some slight curvature. It turns out that this curvature, however slight, can cause the revenue adequacy property to fail.

V. Congestion Revenues, FTRs, and Convexity

It is illustrative to continue our example to demonstrate that revenue adequacy is not guaranteed when the feasible set is not convex, even though it may be close to convex.

Consider the case in which the generators at buses 1 and 2 offer substantially different bids in an energy market. With no loss of generality, let us set the offer price at bus 1 to be a constant \$10/MWh and the offer price at bus 2 to be a constant \$20/MWh. (The reader may verify that similar results are obtained by reversing the offer prices.) Now we examine how the optimal dispatch and LMPs change as the active power load is increased at bus 3. We refer to Fig. 5 for this discussion. Low levels of load are supplied by the lower-cost generator at bus 1 and the LMPs are the same and equal to \$10/MWh. In Fig. 5, this dispatch corresponds to the horizontal axis between points O and A. At a load of 8.21 MW, at point A, the capability of the transmission network to deliver energy from bus 1 to bus 3 becomes constrained by the capacity limit of the line connecting buses 1 and 2. To meet demand greater than 8.21 MW requires some supply from the higher-cost generator at bus 2. The slope of the tangent line at point A, $m = 3.54$, graphically illustrates the relative proportion of incremental supply that will come from generators 1 and 2. This can be used to calculate the locational marginal cost at bus 3:[6]

$$\lambda_3 = \frac{3.54 * \left(\frac{\$20}{\mathrm{MWh}}\right) + 1 * \left(\frac{\$10}{\mathrm{MWh}}\right)}{3.54 + 1} = \frac{\$17.80}{\mathrm{MWh}}.$$

[6]Of course, this is typically computed from the optimization routine. We present the graphical approach here to provide intuition for the result.

The LMPs at buses 1 and 2 are equal to the generator offer prices $\lambda_1 = \$10/\mathrm{MWh}$ and $\lambda_2 = \$20/\mathrm{MWh}$, respectively. Using these energy prices, the payment for energy by the load at bus 3 equals $(8.21\ \mathrm{MW}) * (17.80\$/\mathrm{MWh}) = \$146.14/\mathrm{h}$, while the payments received by the generators equals $(8.21\ \mathrm{MW}) * (10.00\$/\mathrm{MWh}) = \$82.10/\mathrm{h}$. The premium charged to the load due to congestion is the total congestion revenue and equals the difference between the load payment and generators' receipts: \$64.04/h.

We have shown earlier that the set of feasible power injections is not necessarily convex. We complete the example by determining the effect this has on revenue adequacy. We will see that "close" to convex does not suffice to ensure revenue adequacy.

In our example with the feasible set of power injections shown in Fig. 5, an auction of FTRs will likely settle to one of the extreme points labeled on the plot: A, B, or C. Without loss of generality, we assume that to maximize income from the FTR auction, it settles to quantities represented at point B: 11.16 MW for the bus 1 POI and bus 3 POW and 7.5 MW for the bus 2 POI and bus 3 POW.[7] (Choosing point A or point C instead would make little difference, as we discuss later.) Evaluating the FTR payments for an operating point at A, with the LMPs calculated earlier, the total FTR payments will be

$$(11.16\ \mathrm{MW}) * \frac{(17.80\text{–}10.00)\$}{\mathrm{MWh}} + (7.5\ \mathrm{MW}) * \frac{(17.80\text{–}20.00)\$}{\mathrm{MWh}} = \frac{\$70.55}{\mathrm{h}}$$

which exceeds the collected congestion revenues of \$60.04/MWh by more than 10%. Revenue adequacy fails.

A complete analysis of this example shows that despite a close to convex feasibility region, revenue adequacy fails at every constrained operating point between points A and B and most points between C and B. If the auction had settled to point A instead, revenue adequacy would still fail for all operating points between A and B. If the auction had settled to point C (unlikely with the expected prices in this case), revenue adequacy would fail for most points between C and B. "Close" to convex is not a sufficient argument to declare revenue adequacy as a property of a system. In our example, the feasible region is arguably close to convex, but revenue adequacy almost always fails.

VI. Conclusions

This paper has focused on the study of convexity of the space of feasible power injections that satisfy the power flow equations subject to a number of capacity and voltage constraints. We have proven that in general, the feasibility region will not be convex and have shown by example that the projection on the space of active powers is also not guaranteed to be convex. We have examined the consequence of a nonconvex feasibility region on revenue adequacy in FTR markets. Importantly, we note that "close" to convex is not sufficient to provide revenue adequacy. While the boundaries of the feasibility region may be nearly linear, the curvature, however slight, will determine whether the

[7]In this case, it is possible that FTR auction participants will pay to obtain FTRs from bus 1 to 3 but will be paid to take FTRs from bus 2 to bus 3.

system will be revenue adequate. We have also provided detailed small examples in which we demonstrate nonconvexity and revenue inadequacy. It is clear that these same problems can arise in large systems, and in our future work, we will investigate models for such systems. Here we have focused on theory.

In practice, ISO markets have not proven to be revenue adequate at all times. For example, the NYISO consistently reports congestion revenue shortfalls during the summer [22], [23] and annual shortfalls of $49M, $77M, and $126M in day-ahead congestion rents in 2001, 2002, and 2003, respectively [24]. To cope with these shortfalls, the NYISO implemented polices in 2003 and 2004 to assign financial coverage of shortages and benefit of surpluses to transmission owners and to allow transmission owners to reserve a small portion of transmission from TCC auctions to try to avoid congestion revenue shortages [24].

The dominant cause for inadequacy has not been completely studied, to the authors' knowledge. In addition to a possibly nonconvex feasibility region, it is also the case that the actual network (topology) may differ from that assumed in the simultaneous feasibility test. In their yearly report [22], NYISO asserts that the day-ahead congestion revenue shortfalls are "largely due to transmission outages that are reflected in the day-ahead market but are not included in the TCC auction." In [25], Liu and Gross studied the effect of parameter uncertainties and line outages on revenue adequacy using a convex, dc power flow approximation. They conclude that the effects are minor and that "over a given period, it is reasonable to expect that there is revenue adequacy." We suggest here that the use of an ac power flow (as is used by NYISO[8]) may accentuate the effect of unplanned line outages to create congestion revenue shortfalls or be the cause of such shortfalls.

In [14], Oren and Ross warned against a proposed policy for congestion relief that suffered from a misapplication of a Lagrangian relaxation technique that could, in some cases, result in a suboptimal or infeasible outcome. In this paper, we express our concern that unproven assumptions about system (convexity) properties may be used to assert other properties (revenue adequacy) that are used to justify policy. Since commencing operation of FTR markets, ISOs have appropriately implemented policies to handle situations in which congestion revenues fall short of FTR obligations. Electricity markets are young and proposals for new FTR mechanisms are still being proposed and examined. For example, in [11], a system of obligations and options is proposed and revenue adequacy is proven assuming a linear dc power flow model. To extend the revenue adequacy result to an ac power flow model would likely require an assumption about convexity of the feasible set of power flow injections.

As market designs evolve and improve, it is reasonable to expect that congestion management tools will tend to employ the most accurate models possible, and dc power flows will give way to more accurate ac power flow models. Unfortunately, as these more detailed models gain wider acceptance, it will be impossible to prove revenue adequacy or expect it in practice. Adoption of new FTR mechanisms will need to be accompanied by policies for accommodating congestion revenue shortfalls.

[8]The NYISO is presently the only ISO that uses a full ac power flow.

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