Power System Applications of Trajectory Sensitivities

Ian A. Hiskens, *Senior Member* M.A. Pai, *Life Fellow*
Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign
Urbana IL 61801 USA

Abstract— **Trajectory sensitivities complement time domain simulation in the analysis of large disturbance dynamic behaviour of power systems. They are formed by linearization around a nonlinear, and possibly non-smooth, trajectory. The influence of parameter variations on large disturbance behaviour can be estimated (to first order) from these sensitivities. Large (small) sensitivities indicate that a parameter has a significant (negligible) effect on behaviour. These insights are helpful in analysing the underlying influences on system dynamics, and for assessing the significance of parameter uncertainty. Further, trajectory sensitivities provide gradient information for applications such as parameter estimation, boundary value problems, border-collision bifurcations, and optimal control.**

Keywords: Power system dynamics, trajectory sensitivities, parameter estimation, inverse problems.

I. Introduction

TOOLS for systematically exploring large disturbance behaviour of power systems are quite limited. Generally such analysis relies on time domain simulation and intuition. Whilst simulation provides a vast amount of information on the behaviour of system states, it fails to provide insights into underlying parametric influences. The intuition of experienced analysts can compensate for those deficiencies, however many organizations are suffering a decline in that experience base.

Further, analysis of power system dynamics is model based, implying reliance on an accurate knowledge of parameter values. Inaccurate parameters can lead to incorrect (and possibly expensive) conclusions. However determining the sensitivity of behaviour to parameters has traditionally relied on repeated simulation, which is extremely time consuming.

Recent work has shown that trajectory sensitivity analysis can provide valuable insights into the security of power systems [1], [2]. This paper proposes the use of trajectory sensitivities to complement time domain analysis of power system dynamics.

The influence of parameters on the nonlinear, non-smooth behaviour exhibited by a disturbed power system is difficult to explore. Normal linearization techniques, involving linearization of the system model about an operating point, are not applicable. However trajectory sensitivity analysis offers a rigorous approach to exploring the effects of parameters [3]. This analysis is based on linearizing the system around a trajectory, rather than around an equilibrium point [4], [5]. Therefore it is possible to determine directly the change in the trajectory due to a (small) change in parameters. The ideas extend naturally through discontinuities, provided a few technical conditions are satisfied [3].

The paper is structured as follows. An overview of trajectory sensitivity concepts is provided in Section II. Numerous application are outlined in Section III, and conclusions are presented in Section IV.

Research supported by the EPRI/DoD Complex Interactive Networks/Systems Initiative, the National Science Foundation through grant ECS-0114725, and the Grainger Foundation.

II. Trajectory Sensitivity Concepts

Power system dynamic behaviour often exhibits interplay between continuous dynamics and state-driven discrete events. Such behaviour can be captured by a differential-algebraic model that incorporates impulsive action and switching (DAIS model). That model is fully described in [3], [6]. However to facilitate a clearer presentation of trajectory sensitivity concepts, this paper will use the simpler model

$$\dot{x} = f(x), \qquad x(t_0) = x_0. \tag{1}$$

Parameters λ can be incorporated through trivial differential equations

$$\dot{\lambda} = 0, \qquad \lambda(t_0) = \lambda_0. \tag{2}$$

It is convenient to describe the response of the model (1) in terms of the *flow* of x, defined as

$$x(t) = \phi(x_0, t) \tag{3}$$

where $x(t)$ satisfies (1), including the initial conditions $\phi(x_0, t_0) = x_0$.

Trajectory sensitivities provide a way of quantifying the variation of a trajectory (flow) resulting from (small) changes to parameters and/or initial conditions [3], [4]. To obtain the sensitivity of the flow ϕ to initial conditions x_0, the Taylor series expansion of (3) is formed, and higher order terms are neglected,

$$\begin{aligned} \Delta x(t) &= \frac{\partial \phi}{\partial x_0}\Delta x_0 + \text{ higher order terms} \\ &\approx \frac{\partial x(t)}{\partial x_0}\Delta x_0 \equiv x_{x_0}(t)\Delta x_0. \end{aligned} \tag{4}$$

By incorporating parameters via (2), sensitivity to initial conditions x_0 includes parameter sensitivity. Equation (4) describes the change $\Delta x(t)$ in a trajectory, at time t along the trajectory, for a given (small) change in initial conditions Δx_0. The time-varying partial derivatives x_{x_0} are known as *trajectory sensitivities.*

The actual sensitivities $x_{x_0}(t)$ are obtained by differentiating (1) with respect to x_0, giving

$$\dot{x}_{x_0} = f_x(t) x_{x_0} \tag{5}$$

where $f_x \equiv \partial f / \partial x$ is a time-varying Jacobian matrix that is evaluated along the trajectory. Initial conditions are obtained from (3) at t_0 as

$$x_{x_0}(t_0) = I$$

where I is the identity matrix.

For large systems, the linear time-varying equations (5) have high dimension. However the computational burden is minimal when an implicit numerical integration technique such as trapezoidal integration is used to generate the trajectory. More complete details are given in [3], [7].

The development of trajectory sensitivities for the more complete DAIS model is conceptually similar [3]. The technical details are more intricate however.

III. Applications

A. Parametric influences on system dynamics

Analysis of large disturbance dynamic behaviour often raises questions relating to the influence of parameters throughout the event. For example, investigations of a disturbance on the Nordel system [8] considered (among other things) the relative influence of certain lines on behaviour. Useful insights were obtained from the sensitivity of the angle trajectory to line impedances.

This example illustrates the types of insights that can be provided by trajectory sensitivities, and which are not available (directly) from simulation. Sensitivities of any system state trajectory to any parameter (even parameters describing hard nonlinearities such as limits) can be obtained for negligible extra computational cost. The participation of components in an event can be quickly assessed.

B. Parameter uncertainty

System parameters can never be known exactly. In fact uncertainty in some parameters, e.g., load models, can be quite high. Quantifying the effects of parameter uncertainty illustrates the usefulness of trajectory sensitivities.

Because of the uncertainty in parameters, investigation of system behaviour should (ideally) include multiple studies over a range of parameter values. However simulation of large systems is computationally expensive. Such an investigation would be extremely time consuming. The generally adopted approach is to assume that a *nominal* set of parameters provides an adequate representation of behaviour over the full range of values. This may not always be a good assumption though.

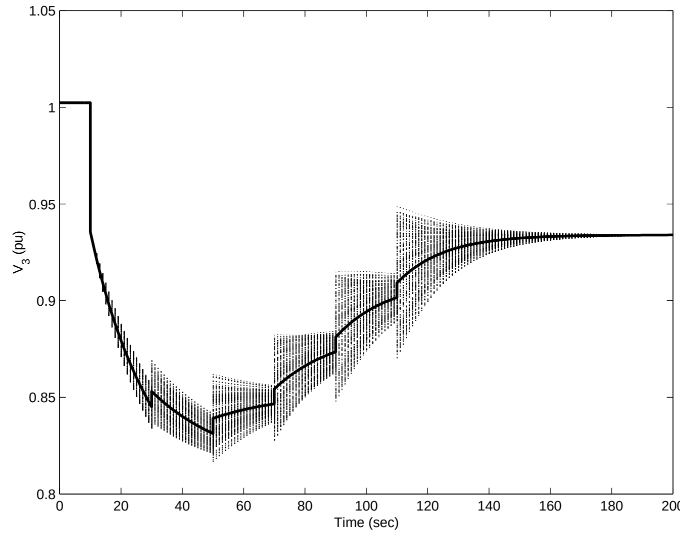

Fig. 1. Trajectory bounds.

A computationally feasible (though approximate) alternative to repeated simulation is to generate a first order approximation of the trajectory for each set of perturbed parameters. The first order approximation is obtained from the truncated Taylor series expansion of the flow ϕ. Using (4) gives

$$\phi(x_{0_2}, t) \approx \phi(x_{0_1}, t) + x_{x_0}(t)(x_{0_2} - x_{0_1}) \tag{6}$$

where $x_{x_0}(t)$ is computed along the nominal trajectory $\phi(x_{0_1}, t)$. Therefore if the trajectory sensitivities $x_{x_0}(t)$ are available for a nominal trajectory, then (6) can be used to provide a good estimate of trajectories $\phi(x_{0_2}, t)$ corresponding to other (nearby) parameter sets. (Recall that parameters λ are embedded in x_0.)

The computational burden involved in generating the approximate trajectories is negligible. Given the nominal trajectory and associated trajectory sensitivities, new (approximate) trajectories can be obtained for many parameter sets. Therefore a Monte-Carlo technique can be employed to quantify the uncertainty in a trajectory:

- parameter sets are randomly generated,
- first order approximations are obtained using (6).

Figure 1 illustrates this process for a simple example where a disturbance initiated interactions between a tap-changing transformer and a dynamic load. The dark line shows the nominal trajectory. The bound around that trajectory was obtained using 200 randomly chosen sets of parameters. Further details can be found in [9].

Statistics quantifying the uncertainty in system behaviour due to parameter uncertainty can be obtained from the Monte-Carlo simulation. For example, it's possible to state the probability that a disturbance would initiate protection operation or that a voltage would fall below some predetermined threshold.

Another approach to assessing the significance of parameter uncertainty is via worst case analysis [10]. This involves finding the values of parameters (within specified bounds) that induce the greatest deviation in particular system variables, for example voltages. The algorithm can be formulated as a constrained optimization. Similar optimization problems are discussed in Section III-D.

C. Parameter estimation

System-wide measurements of power system disturbances are frequently used in event reconstruction to gain a better understanding of system behaviour [11], [12]. In undertaking such studies, measurements are compared with the behaviour predicted by a model. Differences are used to tune the model, i.e., adjust parameters, to obtain the best match between the model and the measurements. This process requires a systematic approach to,

1) identifying *well-conditioned* parameters that can be estimated reliably from the available measurements, and
2) obtaining a best estimate for those parameters.

It is shown in [13] that trajectory sensitivities can be used to guide the search for well-conditioned parameters, i.e., parameters that are good candidates for reliable estimation. Large trajectory sensitivities imply the corresponding parameters have leverage in altering the model trajectory to better match the measured response. Small trajectory sensitivities, on the other hand, imply that large changes in parameter values would be required to significantly alter the trajectory. Parameters in the former category are well-conditioned, whereas the latter parameters are ill-conditioned. Only parameters that influence measured states can be identified. A parameter may have a significant influence on system behaviour, but if that influence is not observable in the measured states, then the parameter is not identifiable. The concept of identifiability is explained more formally in [14].

The use of trajectory sensitivities in parameter estimation is not new [4]. In the power systems context, similar ideas have been used for estimating parameters of generators and AVRs/exciters [15], [16], [17], [18]. In fact, the estimation process has been adapted to model reduction [19]. The number of parameters that could be estimated using those earlier ideas was limited though, because trajectory sensitivities were generated numerically [17], [18]. By exploiting more computationally efficient methods of calculating trajectory sensitivities [3], it is possible to consider many system-wide parameters.

A parameter estimation algorithm that is based on a Gauss-Newton iterative procedure is presented in [13]. The algorithm minimizes the nonlinear least-squares cost

$$\mathcal{V}(\theta) = \frac{1}{2}\|\breve{x}(\theta) - ms\|_2^2$$

where ms are the sampled measurements of the disturbance, $\breve{x}(\theta)$ are the flows provided by the model that correspond to the measured quantities, and θ are the unknown parameters. This minimization can be achieved (locally at least) by the iterative scheme

$$S(\theta^j)^t S(\theta^j)\Delta\theta^{j+1} = S(\theta^j)^t(\breve{x}(\theta^j) - ms) \qquad (7)$$
$$\theta^{j+1} = \theta^j - \alpha^{j+1}\Delta\theta^{j+1}$$

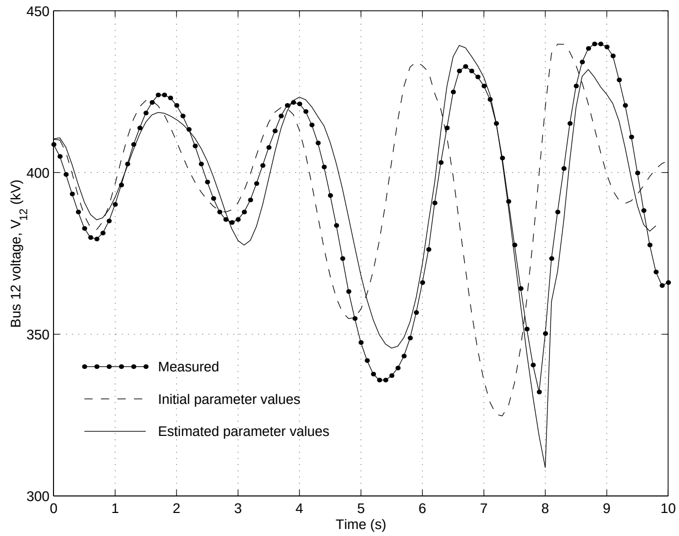

Fig. 2. Parameter estimation.

where α^{j+1} is a scalar that determines the parameter update step size[1]. The matrix S is built from the trajectory sensitivities $\breve{x}_\theta$, i.e., sensitivity of model flows $\breve{x}$ to parameters θ. The invertibility of S^tS relates directly to identifiability [14].

The parameter estimation process is illustrated in Figure 2. A voltage measurement from a disturbance on the Nordel system [13] is shown. The figure also shows the simulated voltage trajectory for the initial parameter values, and the tuned values obtained after convergence (in four iterations) of (7). The improvement is clear.

Remarks:

1) Parameter estimation via (7) is not restricted to smooth systems. In fact, it is possible to estimate parameters that underlie event descriptions (provided measurements capture an occurrence of the event.)
2) For large systems, feasibility of the Gauss-Newton algorithm is dependent upon efficient computation of trajectory sensitivities. This underlines the importance of systematic modeling.

D. Inverse problems

System analysis is often tantamount to understanding the influence of parameters on system behaviour, and applying that knowledge to achieve a desired outcome. The 'known' information is the desired outcome. The parameters that achieve that outcome must be deduced. Parameter estimation, as described in the previous section, is a classic example [21].

Because of the inverse nature of such problems, the process has traditionally involved repeated simulation of the model. This can be time consuming and frustrating, as the relationship between parameters and behaviour is often not intuitively obvious.

[1]Equation (7) could be solved by inverting S^tS, however faster and more numerically robust algorithms are available [20].

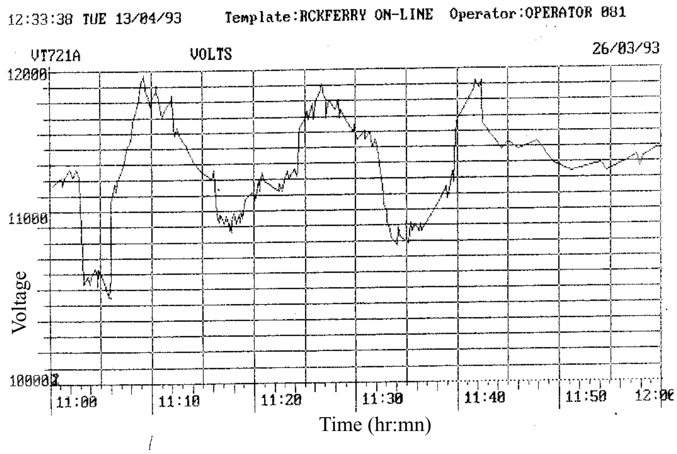

Fig. 3. Oscillations in distribution system voltage.

Systematic modeling allows the development of new tools that can solve inverse problems directly, albeit via iterative techniques. The DAIS model [6] is conducive to the efficient generation of trajectory sensitivities. They underlie the development of gradient-based algorithms.

The following subsections present a range of inverse problems. Algorithms that address those problems are outlined. This list is not exhaustive, but seeks to provide an overview of the possibilities.

D.1 Boundary value problems

Boundary value problems *per se* are uncommon in power systems. However an application of increasing importance is the calculation of limit cycles (sustained oscillations). Oscillations have been observed in a variety of power systems, from generation [22] to distribution. This latter case is illustrated in Figure 3, where the oscillations were driven by interactions between loads, transformer tapping and capacitor switching.

Boundary value problems take the form

$$r\big(x_0, x(t_f)\big) = 0 \tag{8}$$

where t_f is the final time, and $x(t)$ is the trajectory that starts from x_0 and is generated by (1). The initial values x_0 are variables that must be adjusted to satisfy r. (Though r may directly constrain some elements of x_0.) To establish the solution process, (8) can be rewritten

$$r\big(x_0, \phi(x_0, t_f)\big) = 0, \tag{9}$$

which has the form $\tilde{r}(x_0) = 0$. Boundary value problems can be solved by shooting methods [23], [24], which are a combination of Newton's method for solving (9) along with numerical integration for obtaining the flow ϕ. Newton's method requires the Jacobian

$$J = \frac{\partial r}{\partial x_0} + \frac{\partial r}{\partial \phi_x} x_{x_0}(t_f), \tag{10}$$

which is dependent upon the trajectory sensitivities evaluated at t_f.

To solve for limit cycles, (9) can be reformulated as

$$x_0 - \phi_x(x_0, T) = 0$$

where x_0 lies on the limit cycle and T is its period. Solution of this boundary value problem via a shooting method requires $x_{x_0}(T)$, which is exactly the Monodromy matrix [23], [25]. The eigenvalues of this matrix determine the stability of the limit cycle.

D.2 Border collision bifurcations

When a system trajectory encounters the operating characteristic of a protection device, a trip signal is sent to circuit breakers. If the trajectory almost touches the operating characteristic but just misses, no trip signal is issued. The bounding (separating) case corresponds to the trajectory grazing, i.e., just touching, the operating characteristic but not crossing it. This is a form of global bifurcation; it separates two cases that have significantly different outcomes. Numerous names exist for this phenomena, including C-bifurcation, switching-time bifurcation and border-collision bifurcation.

Examples of such bifurcations can be found in many other application areas. They are particularly important in power electronic circuits, where zero-crossings are fundamental to control strategies, and to the switching of self-commutating devices [26]. In fact it has been shown that border-collision bifurcations can provide a path to chaos in simple DC-DC converters [27].

Identifying the critical values of parameters that correspond to a border-collision bifurcation is an inverse problem. Let the operating/switching characteristic be described by $b(x) = 0$. A trajectory will be tangential to that characteristic at the point $x^* = \phi(x_0^*, t^*)$ given by

$$b(x^*) = 0$$
$$b_x\big|_{x^*} f(x^*) = 0.$$

The critical values of parameters are given by x_0^*. This is a special form of boundary value problem. Gradient-based algorithms can be established to solve this problem.

Knowledge of the critical parameter values can be used in security assessment to determine the likelihood of contingencies initiating undesirable protection operation.

D.3 Optimal control

Optimization problems arise frequently in the analysis of power system dynamics. Examples range from tuning generator AVR/PSSs to determining the optimal location, amount and switching times for load shedding [28], [29]. All these problems can be formulated using a Bolza form of objective function

$$\min_{\theta, t_f} \mathcal{C}(x, y, \theta, t_f) \tag{11}$$

where

$$\mathcal{C} = \varphi\big(x(t_f), y(t_f), \theta, t_f\big) + \int_{t_0}^{t_f} \psi\big(x(t), y(t), \theta, t\big) dt,$$

θ are the design parameters, i.e., the parameters adjusted to achieve the objective, and t_f is the final time.

The solution of (11) may be complicated by discontinuous behaviour at events. However these complications largely disappear under the assumption that the order of events does not change as θ and t_f vary, i.e., no border-collision bifurcations occur. This assumption is common throughout the literature, though it is expressed in various ways: transversal crossings of triggering hypersurfaces are assumed in [30], existence of trajectory sensitivities is assumed in [31], and [32] assumes all flows have the same *history*. All statements are equivalent.

Under that assumption, and other mild assumptions, it is concluded in [32] that if $\mathcal{C}$ is continuous in its arguments then a solution to (11) exists. Further, [31] shows that if $\mathcal{C}$ is a smooth function of its arguments, then it is continuously differentiable with respect to θ and t_f. The minimization can therefore be solved using gradient-based methods. Trajectory sensitivities provide the gradient information.

If the event ordering assumption is not satisfied, $\mathcal{C}$ may be discontinuous. The optimization problem then takes on a combinatorial nature, as each continuous section of $\mathcal{C}$ must be searched for a local minimum.

Other optimization problems do not naturally fit the form (11) of the objective function. Cascaded tap-changing transformers provide an interesting example [33]. Minimizing the number of tap change operations is equivalent to minimizing the number of crossings of triggering hypersurfaces. Such a problem, by definition, does not satisfy the earlier assumption requiring constant ordering of events. This minimization is best addressed using switching control design techniques [34], though the solution process is not yet well established.

E. Stability assessment

The stability margin for a particular disturbance can be thought of as the smallest "distance" between the system trajectory and the stability boundary. A large margin indicates the system is very stable (for that disturbance), whereas a margin of zero implies impending instability[2]. It has been observed [36], [37], and can be mathematically justified, that as the stability margin reduces, trajectory sensitivities undergo larger excursions. Further, for unstable cases, trajectory sensitivities increase much more rapidly than the underlying system trajectory.

It follows that rapid growth in trajectory sensitivities can be associated with an underlying stability problem. Sensitivities can therefore be used as an early indicator of impending instability. This forms the basis for a filtering process that can rapidly separate the critical dynamic contingency cases from those that are uninteresting.

Trajectory sensitivities can also be used to predict critical values of parameters [38], i.e., values that (theoretically) drive the system trajectory onto the stability boundary. In this application, sensitivities are generated for two values of the parameter of interest. A factor η is calculated as the inverse of the maximum deviation in the sensitivities. As indicated above, the peak values of the trajectory sensitivities are inversely related to the stability margin. Therefore it should be expected that η will approach zero at the critical parameter value. This relationship is illustrated in Figure 4. It has been observed across numerous examples that the $\lambda - \eta$ relationship is almost linear. This allows accurate predictions of critical parameter values. The justification for this linearity is not clear though, and is a focus of on-going research.

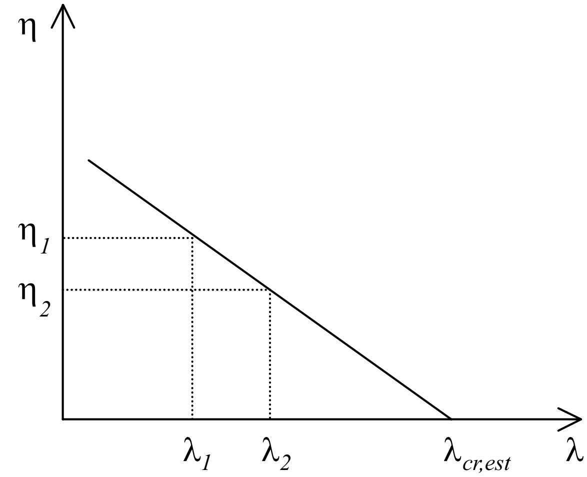

Fig. 4. Prediction of critical parameter value.

[2]Energy function methods have been commonly used to estimate the stability margin [35].

IV. Conclusions

Analysis of large disturbance dynamic behaviour of power systems is largely reliant on time domain simulation. However underlying parametric influences cannot easily be deduced from such analysis. The paper therefore proposes the use of trajectory sensitivities to complement time domain analysis. Trajectory sensitivities can be obtained as a by-product of implicit numerical integration techniques. They incur little additional computational cost.

Trajectory sensitivities can be used directly to provide insights into the influence of parameters on system behaviour. They also facilitate efficient assessment of the significance of parameter uncertainty. Further, trajectory sensitivities provide gradient information that underlies a number of inverse problems, including parameter estimation, boundary value problems, border collision bifurcations, and optimal control.

Trajectory sensitivities are described by their own dynamic system. Their behaviour can provide an early indication of impending system instability. This property also motivates an approach to determining critically stable parameter values.

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