Continuation Techniques for Reachability Analysis of Uncertain Power Systems*

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***Abstract*—The level of uncertainty in power systems is increasing due to substantial growth in renewable generation and increasing reliance on cyber-enabled system-wide responsiveness. In order to guarantee dynamic security of networks, power system designers are in need of practical reachability analysis tools that allow them to assess vulnerability of the system to undesirable outcomes and evaluate the impact of parameter uncertainty. This paper describes a novel suite of parameter continuation algorithms that may be applied to large-scale, stiff, hybrid power systems. They allow the impact of parameter uncertainty to be quantified, thereby enabling secure design through efficient exploration of critical parameter values and initial conditions.**

I. Introduction

The dynamic behavior of power systems is affected by many parameters, whose influence cannot be easily quantified. It is well known, for example, that in many power systems, the behavior of loads can have a significant influence on dynamic performance. Substantial work has been undertaken, both in industry and the research community, to develop methods of quantifying load models. Yet when major disturbances occur, more often than not the match between the measured and simulated responses is substandard, with poorly validated load models being a major contributing factor.

Load models can never be known precisely, as load composition is continually varying. Likewise, as renewable generation grows, its inherent variability will challenge the usefulness of traditional approaches to dynamic security assessment that currently rely almost exclusively on forward simulation.

The concept of *grazing* is important in quantifying the influence of parameters on the dynamic behavior of power systems. Grazing refers to the situation where a state-space trajectory tangentially encounters a hypersurface that has significance in the context of system dynamic performance. As shown in [1], grazing situations dictate the boundary between acceptable and unacceptable behavior. Grazing phenomena may thus form the basis for reachability analysis that can be used to assess the vulnerability of power systems to undesirable outcomes. Computational algorithms can be applied to obtain approximate covers of implicitly defined manifolds in state/parameter space corresponding to grazing-related boundaries between safe and unsafe operation. The analysis of these boundaries provides an opportunity to quantify the dependence of dynamic security on parameter variations.

II. Reachability analysis of power systems

A. Hybrid System Representation

Power system models belong to the class of hybrid dynamical systems [2], in which continuous-in-time changes in the state may be characterized by a family of distinct state-space flows, and discrete-in-time changes in the state (or the governing equations) are triggered by zero crossings of event surfaces and associated with the application of state-space resets. Such systems can be represented by a model that consists of a system of differential-algebraic equations of the form

$$\dot{x} = f(x, y) \tag{1}$$

$$0 = g(x, y) \tag{2}$$

adapted to incorporate impulsive (state reset) action and switching of the algebraic equations. We refer to $x \in \mathbb{R}^n$ as dynamic states, to $y \in \mathbb{R}^m$ as algebraic states, and assume that $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$. Here, the governing functions f and g may change along a time history, as dictated by the physical model.

Under the assumption that the Jacobian $\partial_y g$ is invertible, it is possible to reduce the differential-algebraic equations locally to a system of ordinary differential equations in the dynamic states only. It follows that the dynamic behavior can be described analytically by the *flow* ϕ, i.e., $x(t) = \phi(x_0, t)$ where the time-dependence of the algebraic states is implicitly defined by the algebraic constraints $g(\phi(x_0, t), y(t)) = 0$.

B. Grazing

Grazing is characterized by a tangential encounter of a state-space trajectory with a performance constraint, characterized by a zero-level surface of some function $h : \mathbb{R}^{n+m} \to \mathbb{R}$ (cf. Fig. 1). At such a point of tangential contact, it follows that $h(x, y) = 0$ and, in terms of the gradients of h with respect to x and y,

$$h_x \cdot f + h_y \cdot v = 0 \tag{3}$$

where v is implicitly defined by differentiation of (2) to obtain

$$g_x \cdot f + g_y \cdot v = 0. \tag{4}$$

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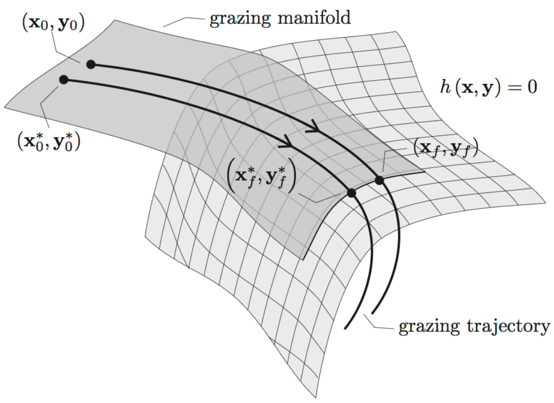

Fig. 1. Grazing manifold.

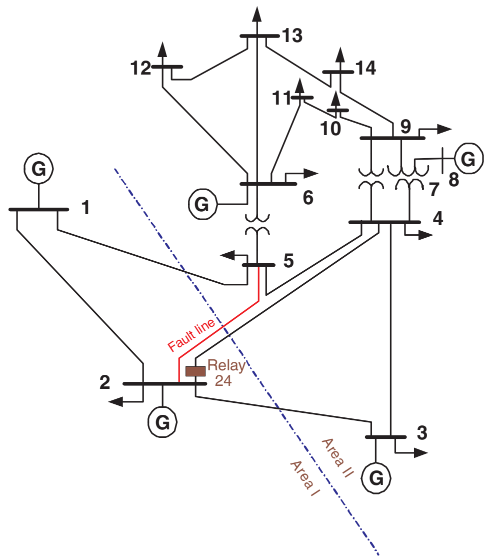

Fig. 2. Modified IEEE 14-bus system.

A multi-segment trajectory with a grazing terminal point may then be characterized in terms of an unknown initial condition x_0, an unknown final condition x_f, the corresponding value y_f, the time of flight t_f, and the unknown vector v that collectively satisfy the zero problem $F = 0$, where the function $F : \mathbb{R}^{2n+2m+1} \to \mathbb{R}^{n+2m+2}$ is given by

$$F(x_0, x_f, y_f, t_f, v) = \begin{pmatrix} \phi(x_0, t_f) - x_f \\ g(x_f, y_f) \\ h(x_f, y_f) \\ h_x \cdot f(x_f, y_f) + h_y \cdot v \\ g_x \cdot f(x_f, y_f) + g_y \cdot v \end{pmatrix} \quad (5)$$

and where the partial derivatives are evaluated at the grazing point (x_f, y_f). Given a regular solution point to the zero problem, i.e., one for which the corresponding Jacobian has full rank, it follows from the implicit function theorem that there exists a locally unique, $(2n+2m+1)-(n+2m+2) = (n-1)$-dimensional manifold of solutions through this point. Each point on this manifold corresponds to a state-space trajectory that terminates at a point of grazing contact with the target surface. This *grazing manifold* thus distinguishes between trajectories that reach the target, and experience the corresponding switching action, and those that do not.

As an example, consider the IEEE 14-bus system of Fig. 2 and the analysis of distance protection on transmission line 2-4 at bus 2 (see [1] for a detailed description of this system). Specifically, consider an operating condition in which a three-phase fault is applied at the midpoint of line 2-5, and the line is removed from service after 0.1 sec, and assume further that relay 24 issues an instantaneous trip signal if the apparent impedance seen along line 2-4 passes into the protection operation circle. The aim of the analysis is then to explore the relationship between values of real power demand at different buses that cause the apparent impedance trajectory to graze the protection hypersurface.

III. Continuation

A. Fundamentals of continuation

The zero problem $F = 0$ with F given in (5) is an example of a general paradigm for implicitly-defined manifolds that serve to characterize a large number of mathematical models [3], [4]. Computational techniques used to locate and locally parameterize families of solutions to such problems are referred to as parameter continuation methods. In the general formalism, an equation of the form $F(u) = 0$, for some vector-valued function $F : \mathbb{R}^a \to \mathbb{R}^b$ in terms of a vector of continuation variables $u \in \mathbb{R}^a$, provides an implicit definition of an $(a-b)$-dimensional manifold, where the *dimensional deficit* $d = a - b$ equals the excess of unknowns to equations.

Starting with a single point on the solution manifold, and a local parameterization of the manifold in terms of coordinates in the corresponding tangent space, a *covering algorithm* provides a growing cover of the manifold by iteratively computing nearby solutions and their associated local parameterizations. The resultant cover provides a formal *atlas* for the manifold, in terms of a collection of manifold *charts*. Covering algorithms are designed to generate simply connected atlases with a minimum of overlap between charts.

The most successful covering algorithm for one-dimensional manifolds is the pseudo-arc length continuation method, for which an initial solution guess for each successive point on the manifold is obtained in the affine space described by a previous point and its tangent space. It is straightforward to implement this algorithm in such a way as to ensure a simply connected atlas with a minimum of chart overlap. Naïve, multi-dimensional generalizations of this paradigm, however, may result in atlases with significant chart overlap and premature termination without guaranteeing simple connectedness. The manifold covering approach developed by Henderson [5] (see an independent implementation in [6]) resolves these challenges by, i) maintaining a distinction between charts in the atlas interior and charts on its boundary, and ii) associating a neighbor relation on the network of charts defined in terms of mutual overlap.

B. An extended continuation problem

The power system example in Section II-B shows an instance of a zero problem, in which a subset of the governing equations are associated with a degenerate intersection between a state-space trajectory and a hypersurface. It is natural to extend the continuation task by considering a relaxed zero problem in which the terminal point is assumed to satisfy all components of the original zero problem except for the requirement that $h(x_f, y_f) = 0$. To this end, we replace the continuation problem by the *extended continuation problem* $\tilde{F} = 0$, for $\tilde{F} : \mathbb{R}^{2n+2m+2} \to \mathbb{R}^{n+2m+2}$ given by

$$\tilde{F}(x_0, x_f, y_f, t_f, v, \mu) = \begin{pmatrix} \phi(x_0, t_f) - x_f \\ g(x_f, y_f) \\ h(x_f, y_f) - \mu \\ h_x \cdot f(x_f, y_f) + h_y \cdot v \\ g_x \cdot f(x_f, y_f) + g_y \cdot v \end{pmatrix}. \quad (6)$$

Here, the *continuation parameter* μ characterizes the signed deviation of the terminal point from the zero-level surface of h.

A solution to the extended continuation problem described by (6) does not necessarily correspond to a grazing trajectory, but such a trajectory may be located by allowing μ to vary during continuation. Specifically, by continuity, a zero-crossing in the value of μ along a curve segment on the n-dimensional solution manifold implies the existence of at least one point along the curve segment for which $\mu = 0$. This point may now be located by an iterative root solver. Once a zero-crossing of μ has been detected and located with desired accuracy, the grazing manifold may be mapped out by restricting the value of μ to 0, thus returning to the restricted continuation problem (5).

C. Multi-segment asynchronous collocation

In practice, the implementation of the first component of F in (5) may be accomplished either by, i) computing $\phi(x_0, t_f)$ through numerical integration of the governing differential-algebraic equations, or ii) discretizing the time histories for the dynamic and algebraic state variables, and imposing suitable boundary and interior conditions to ensure that the discretization is an approximate solution to the governing hybrid dynamical system. The distinction is between a continuation problem in terms of only the initial and terminal values along the solution trajectory and one in which the unknowns provide a finite-dimensional parameterization of the entire state-space trajectory. In the first case, the number of unknowns in the continuation problem is relatively small, but the algorithm requires forward simulation (i.e., successive construction of approximations to points on the state-space trajectory) and may be sensitive to large state-space strains. In the second case, greater robustness to state-space strains is accompanied by a significant increase in the number of unknowns, but without the need to perform any numerical integration. The methods may be considered essentially equivalent by suitable design of the linear solver used by the nonlinear root-finding algorithm in the second case.

In the COCO continuation framework [6], trajectory discretization is implemented in terms of a segment-specific orthogonal collocation approach. This is well-suited to dynamical systems in which individual degrees of freedom are characterized by a small range of distinct time scales. However, for problems with a wide range of governing time scales (the algebraic constraints in power systems provide an extreme case of such a time scale separation), it is appropriate to consider discretization schemes that partition groups of degrees of freedom by their governing time scales and that provide adaptive mesh-generation algorithms to update the discretization during continuation (see [7] for a related discussion of multistep asynchronous splitting integrators used in forward simulation). To this end, a COCO toolbox that supports asynchronous, adaptive trajectory discretization has been developed by the authors in support of general continuation analysis of power systems.

IV. Numerical Example

We provide a numerical illustration of the continuation paradigm described above, and its implementation in COCO, by returning to the IEEE 14-bus system introduced in Section II-B. The implementation relies on a COCO-specific wrapper to the MATLAB-based DAIS model library, developed by one of the authors [8] for simulation and continuation analysis of power systems. In the initial analysis, we perform continuation of solutions to the relaxed extended continuation problem by allowing the real power demand at bus 4 to vary. Continuation produces a one-dimensional solution manifold along which we detect a solution with $\mu = 0$ when the bus 4 demand has increased by 50.4% from its original value of 96.7 MW. To continue the corresponding family of grazing trajectories, we fix the bus 4 load, and impose the constraint that μ equal 0 throughout continuation. A one-dimensional solution manifold results by allowing the real power demand at buses 2 and 5 to vary.

The grazing manifold shown in Fig. 3 separates regions of operation where the distance protection is, or is not, triggered. This is further illustrated by the impedance trajectories in Fig. 5 for three different cases of bus 2 and bus 5 load values. The parameter space locations of these three cases are identified in Fig. 3. A two-dimensional grazing manifold (parameterized by the loads on buses 2, 3 and 5) is shown in Fig. 4.

V. Conclusion

The COCO continuation framework [6] provides an innovative suite of numerical algorithms for analysis and design of hybrid dynamical systems with slow and fast timescales, coupled components, and with state resets and switches typical of power system models. The paper describes the outcome of coupling this continuation framework to an existing library of power system components. This approach supports trajectory discretization methods, based on asynchronous collocation and mesh-adaptation strategies, that enable scaling up to the number of states typical of practical power systems. The

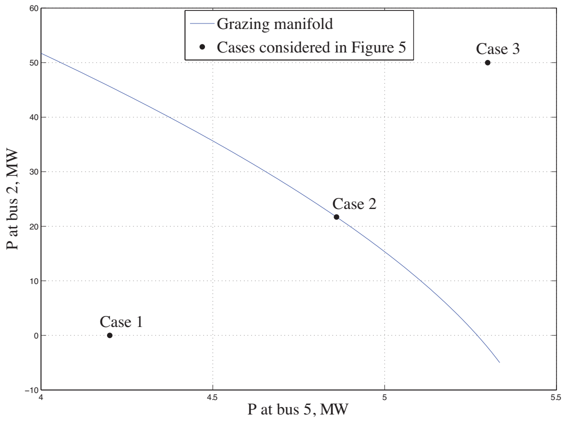

Fig. 3. Curve relating bus 2 and bus 5 loads, produced by continuation.

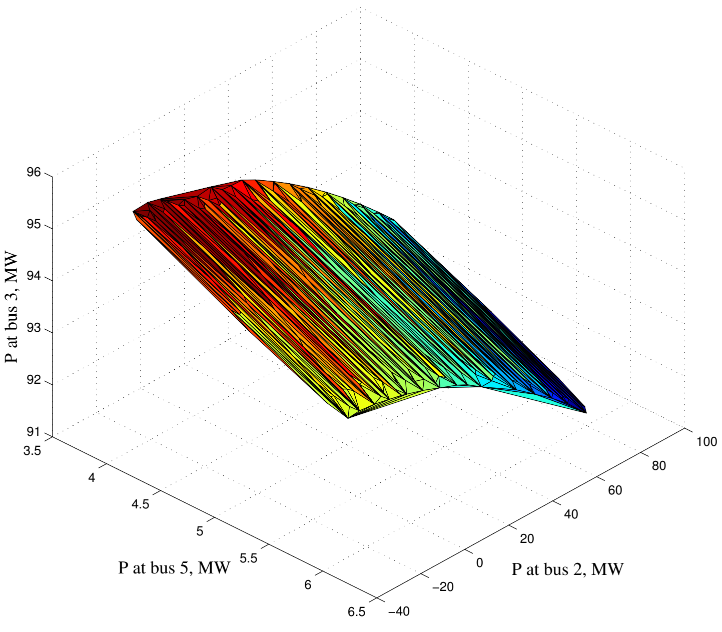

Fig. 4. Grazing manifold relating bus 2, bus 3 and bus 5 loads, produced by continuation.

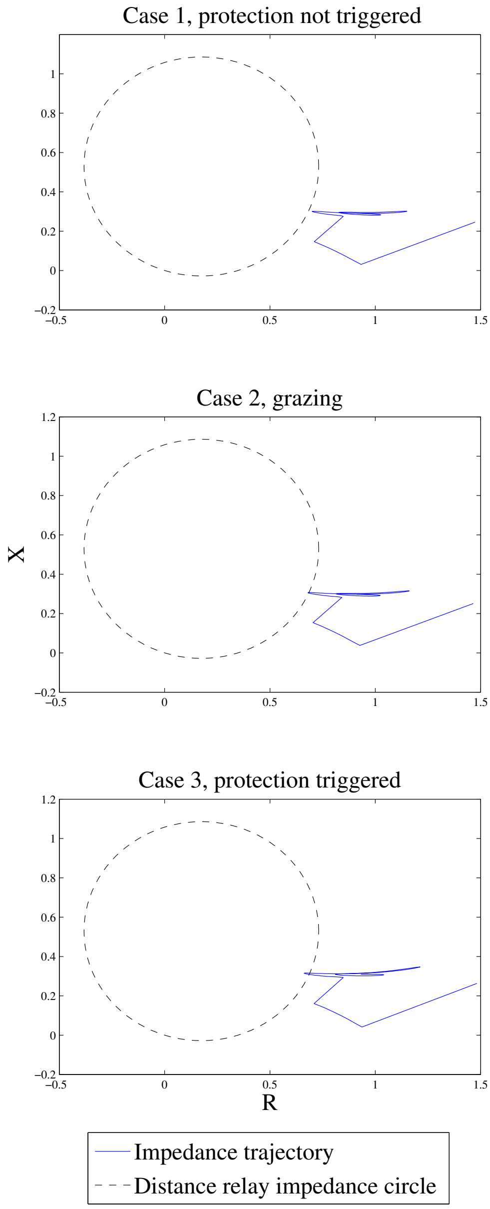

Fig. 5. Power system trajectories in the impedance plane of a distance relay.

coupled framework allows user-driven exploration of switching, threshold and instability related grazing phenomena, and hence provides tools for parametric uncertainty evaluation and vulnerability assessment of networks.

All software algorithms developed as part of this project will be disseminated through an open-source collaborative hosting platform.

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