Trajectory Approximation Near the Stability Boundary*

Ian A. Hiskens
Department of Electrical Engineering and Computer Science
University of Michigan
Ann Arbor, MI 48109 USA

***Abstract*— Uncertain parameters impact the dynamic behaviour of many important nonlinear systems. Assessing the significance of parameter uncertainty is often vitally important, but traditional techniques are computationally challenging. A recent approach uses trajectory sensitivities to form a first-order approximation of perturbed trajectories. These approximate trajectories are generally quite accurate, though their quality reduces in the vicinity of the stability boundary. The paper proposes a modification that improves the accuracy of sensitivity-based approximations of marginally-stable trajectories. This modified approach generates approximate trajectories by considering the intersection of the perturbed trajectory with a hyperplane that progresses with the nominal trajectory.**

I. Motivation

Prediction of system dynamic behaviour is very important for determining whether disturbances may induce instability, and for ensuring performance criteria are satisfied. Uncertainty in parameter values complicates such assessment though. For linear systems, techniques for determining the impact of parameter uncertainty are well developed. Applications abound in control [1] and circuit analysis [2], for example. On the other hand, nonlinear, and especially nonsmooth (hybrid), systems are usually not amenable to such elegant techniques. In such cases, Monte-Carlo techniques often form the basis for assessing the effects of uncertain parameters. For large-scale systems, however, the computational burden of repeating simulations may preclude a thorough investigation of all uncertainties. Important effects may be overlooked.

This issue is very relevant for power systems, where dynamic performance assessment underpins design and operating decisions. Postulated system conditions and disturbance scenarios are investigated to ensure adequate dynamic response. Unfortunately, when actual system events occur, post-mortem analysis invariably reveals discrepancies between modelled and measured system behaviour [3], [4]. The models always contain erroneous parameters, largely as a result of uncertainty in load composition. Furthermore, looking to the future, uncertainty in power system behaviour will grow as widely dispersed renewable resources become an increasingly important source of electrical energy.

Monte-Carlo techniques for assessing the effects of parameter uncertainty are not feasible for large-scale systems, such as power systems, where many parameters are uncertain. A computationally feasible alternative is to replace actual perturbed trajectories by first-order approximations that are generated using trajectory sensitivities [5]. The concepts underlying this approach are outlined in Section II. It was shown in [6] that the resulting approximate trajectories are generally quite accurate, though the accuracy reduces as the trajectory approaches the stability boundary, and nonlinearities become dominant. This paper focuses on improving the fidelity of approximate trajectories under extreme conditions.

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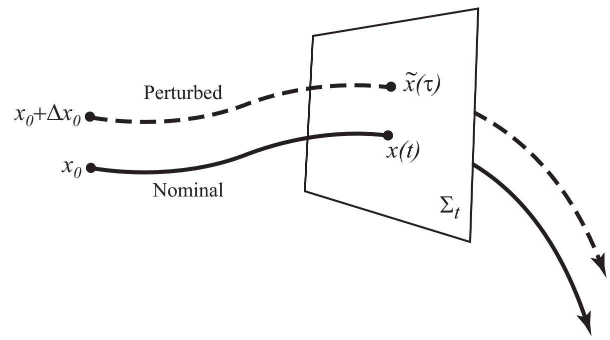

Fig. 1. Time-varying hyperplane concept.

The approach adopted for improving trajectory approximation is developed fully in Section III. The general concepts can be explained with the aid of Figure 1. Consider a hyperplane Σ_t that is anchored to the point $x(t)$ on the nominal trajectory. A perturbation Δx_0 in initial conditions will give rise to a perturbed trajectory that encounters Σ_t at the point $\tilde{x}(\tau)$, where generally $t \neq \tau$. Previous approaches to generating approximate trajectories have focused on the perturbation $x(t) - \tilde{x}(t)$. The approach developed in Section III instead uses the perturbation $x(t) - \tilde{x}(\tau)$ which, by definition, lies along Σ_t.

II. Background

A. Trajectory sensitivities

Trajectory sensitivity concepts are well defined for differential-algebraic systems, and for hybrid dynamical systems [7]. However, for clarity of presentation, this paper will focus on smooth nonlinear systems described by

$$\dot{x} = f(x), \qquad x(0) = x_0, \tag{1}$$

though the subsequent developments extend naturally to the more general setting. Dynamic behaviour generated by (1) can be expressed as the flow,

$$x(t) = \phi(x_0, t) \tag{2}$$

with initial conditions implying $x_0 = \phi(x_0, 0)$.

The functional form of the flow motivates the Taylor series expansion

$$\phi(x_0 + \Delta x_0, t) = \phi(x_0, t) + \frac{\partial \phi(x_0, t)}{\partial x_0}\Delta x_0 + \text{higher order terms.}$$

For small $\|\Delta x_0\|$, the higher order terms may be neglected, giving

$$\begin{aligned}\Delta x(t) &= \phi(x_0 + \Delta x_0, t) - \phi(x_0, t) \\ &\approx \frac{\partial \phi(x_0, t)}{\partial x_0} \Delta x_0 \equiv \Phi(x_0, t) \Delta x_0 \end{aligned} \tag{3}$$

where Φ is the *sensitivity transition matrix*, or *trajectory sensitivities* [5], associated with the flow ϕ. Equation (3) describes the approximate change $\Delta x(t)$ in a trajectory, at time t along the trajectory, for a given small change in initial conditions Δx_0. Note that the sensitivity of trajectories to parameter variations can be incorporated directly into this formulation by introducing trivial differential equations $\dot{x}_p = 0$, with $x_p(0) = p$, where p are the parameters of interest.

The evolution of the trajectory sensitivities Φ is described by variational equations that are obtained by differentiating (1) with respect to x_0. This gives

$$\dot{\Phi} = f_x(t)\Phi, \qquad \Phi(x_0, 0) = I \tag{4}$$

where $f_x(t) \equiv \partial f(t)/\partial x$ is evaluated along the trajectory, and I is the identity matrix. The computational burden of numerically integrating this (potentially high order) linear time-varying system is not particularly onerous, if thoughtfully implemented. It is shown in [7], [8], [9] that when an implicit numerical integration technique such as trapezoidal integration is used, the solution of (4) can be obtained as a by-product of computing the underlying nominal trajectory.

B. Trajectory approximation

For general nonlinear systems, the flow ϕ given by (2), cannot be expressed in closed form. Any change in initial conditions[1] therefore requires a complete re-simulation of the dynamic model. However if changes are relatively small, the computational effort of repeated simulation can be avoided by forming approximate trajectories.

Rearranging (3) gives the first-order approximation of the perturbed flow,

$$\phi(x_0 + \Delta x_0, t) \approx \phi(x_0, t) + \Phi(x_0, t)\Delta x_0. \tag{5}$$

As mentioned previously, trajectory sensitivities Φ can be computed efficiently as a byproduct of simulating the nominal trajectory. Therefore a range of (approximate) perturbed trajectories are available via (5) for the computational cost of a single nominal trajectory.

C. Example

Trajectory approximation concepts will be illustrated using the simple power system example of Figure 2. This system consists of a single generator connected to an "infinite bus". The generator is represented by the classical machine model,

$$\dot{\delta} = \omega \tag{6}$$

$$\dot{\omega} = \frac{1}{M}\Big(P_M - \frac{V_1 V_2}{X}\sin\delta - D\omega\Big) \tag{7}$$

where δ is the generator shaft angle relative to the reference established by the infinite bus, and ω is the generator angular velocity. This model is effectively a damped nonlinear pendulum.

[1]Keep in mind that parameters are incorporated into the initial conditions.

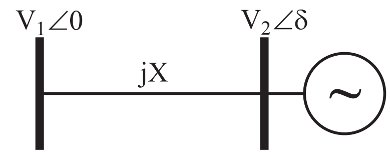

Fig. 2. Single machine infinite bus example.

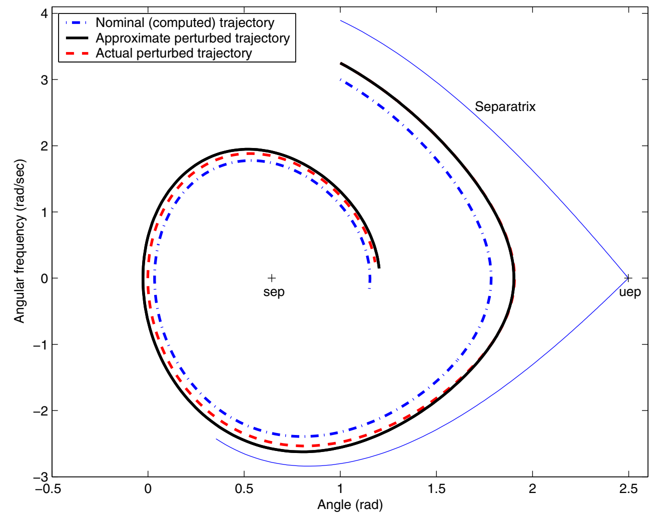

Fig. 3. Phase portrait for initial case, $x_0^\top = [1\ \ 3]$.

It should be mentioned that trajectory approximations have been used in cases that are far more elaborate than (6)-(7). Various examples are provided in [6]. This present example has been chosen as it allows clear illustration of the concepts developed throughout the paper.

Figures 3 and 4 illustrate the application of (5) for generating trajectory approximations. Two cases are considered, and the respective phase portrait plots are provided. In Figure 3, the nominal trajectory is shown as a dash-dot line emanating from the initial point $x_0^\top = [\,\delta(0)\ \ \omega(0)\,] = [\,1\ \ 3\,]$. A perturbation of $\Delta x_0^\top = [\,0\ \ 0.25\,]$ was introduced, and the corresponding approximate trajectory generated using (5). It is shown as a solid line emanating from $(x_0 + \Delta x_0)^\top = [\,1\ \ \ 3.25\,]$. For comparison, the true perturbed trajectory was simulated, and is shown as a dashed line. It can be seen that the approximation is initially very accurate, though a small discrepancy occurs as the trajectory progresses.

Previous investigations, reported in [6], have observed that as trajectories move closer to the boundary of the region of attraction (separatrix), the quality of approximations given by (5) diminishes. As the nonlinearity associated with the separatrix begins to dominate, the influence of the higher order terms, neglected in (5), becomes more significant. The second example considers such a case.

The separatrix is shown in both Figures 3 and 4, along with the corresponding unstable equilibrium point (uep). This second case considered the initial condition $x_0^\top = [\,1\ \ 3.6\,]$, with the nominal trajectory shown in Figure 4 as a dash-dot line emanating from that point. Again, a perturbation of $\Delta x_0^\top = [\,0\ \ 0.25\,]$ was chosen, with the approximate trajectory

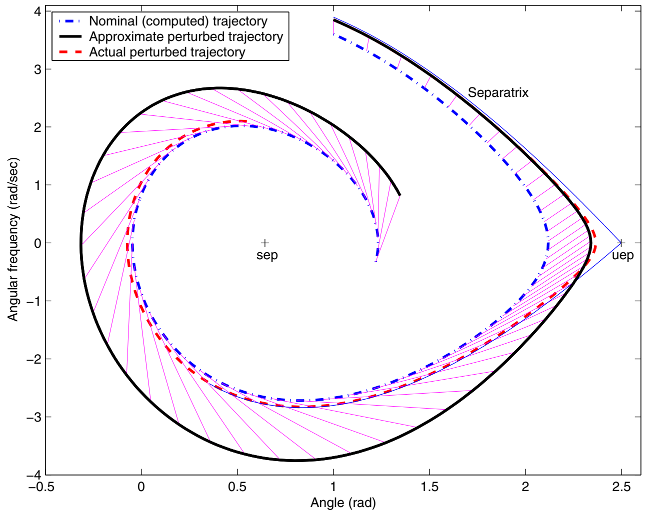

Fig. 4. Phase portrait for highly disturbed case, $x_0^\top = [1\ \ 3.6]$. Traditional trajectory approximation.

shown as a solid line. The dashed line gives the true perturbed trajectory. This perturbed trajectory is in close proximity to the separatrix. Notice in this case that the approximation is quite accurate until it reaches the vicinity of the uep, but then is useless beyond the uep.

To help explain the impact of the uep, Figure 4 includes lines that show the perturbations $\Phi(x_0,t)\Delta x_0$ separating the nominal and approximate trajectories, see (5). Those lines are shown for every fifth time step along the simulation. It is apparent that as the uep is approached, trajectories that are closer to the separatrix proceed more slowly, resulting in a time skew between adjacent trajectories. This stretching of the $\Phi(x_0,t)\Delta x_0$ perturbations underlies the loss of fidelity of the approximate trajectory.

An approach that overcomes this stretching phenomenon is presented in the following section.

III. Trajectory approximation using a sliding hyperplane

A. Conceptual background

The accuracy of approximations generated by (5) suffers when the nominal and perturbed trajectories proceed at substantially different rates. Under such conditions, time-synchronized points $x(t)$ and $\tilde{x}(t)$, on the nominal and perturbed trajectories respectively, will separate. This will occur even though the trajectories may remain close in state space. The example considered in Section II-C, and in particular the second case shown in Figure 4, fit that situation. When the difference $x(t)-\tilde{x}(t)$ becomes large, the higher order terms that are neglected in (5) become important, and the sensitivity term $\Phi(x_0,t)\Delta x_0$ can no longer provide an adequate approximation.

In order to overcome this difficulty, an approach that exploits the state-space proximity of adjacent trajectories is proposed. Referring back to Figure 1, consider a hyperplane Σ_t that contains the point $x(t)$ on the nominal trajectory, and assume that the nominal trajectory is transversal to Σ_t. The perturbed trajectory will pass through Σ_t at the point $\tilde{x}(\tau)$, where generally $\tau \neq t$. The aim of hyperplane-based trajectory approximation is to use trajectory sensitivities, together with the nominal trajectory, to synthesize an approximation for the point $\tilde{x}(\tau)$. By sliding the hyperplane Σ_t along the nominal trajectory, an approximation can be built for the entire perturbed trajectory.

B. Mathematical formulation

The first step in establishing this trajectory approximation process is to determine the relationship between the perturbed initial conditions $\tilde{x}_0$, and the time $\tau(\tilde{x}_0)$ at which the perturbed trajectory encounters the hyperplane Σ_t. Without loss of generality, move the origin to $x(t)$, and let h_t be a vector orthogonal to Σ_t. Then $h_t^\top y = 0$ if and only if $y \in \Sigma_t$.

Let U be a neighbourhood of x_0. A trajectory that emanates from $\tilde{x}_0 \in U$ will encounter Σ_t at the point $\phi(\tilde{x}_0,\tau)$. Define,

$$g(\tilde{x}_0,\tau) := h_t^\top \phi(\tilde{x}_0,\tau). \tag{8}$$

The encounter point is then given by $g(\tilde{x}_0,\tau) = 0$. The implicit function theorem can be used to obtain $\tau(\tilde{x}_0)$ provided

$$\frac{\partial g}{\partial \tau}(x_0,t) = h_t^\top \dot{\phi}(x_0,t) = h_t^\top f(x(t)) \neq 0, \tag{9}$$

which is true because the trajectory at $x(t)$ is transversal to Σ_t. Therefore a function $\tau(\tilde{x}_0)$ exists, such that for $\tilde{x}_0 \in U$, $g(\tilde{x}_0,\tau(\tilde{x}_0)) = 0$, and therefore $\phi(\tilde{x}_0,\tau(\tilde{x}_0)) \in \Sigma_t$. It also follows that $\tau(x_0) = t$.

Furthermore, differentiating g gives,

$$\frac{\partial g}{\partial x}(x_0,\tau(x_0)) + \frac{\partial g}{\partial \tau}(x_0,\tau(x_0))\frac{\partial \tau}{\partial x}(x_0) = 0. \tag{10}$$

From (8),

$$\frac{\partial g}{\partial x}(x_0,t) = h_t^\top \frac{\partial \phi}{\partial x}(x_0,t) = h_t^\top \Phi(x_0,t),$$

with the second equality following from definition (3). Using this together with (9) allows (10) to be expressed as

$$h_t^\top \Phi(x_0,t) + h_t^\top f(x(t))\frac{\partial \tau}{\partial x}(x_0) = 0,$$

and it follows that

$$\frac{\partial \tau}{\partial x}(x_0) = \frac{-1}{h_t^\top f(x(t))} h_t^\top \Phi(x_0,t). \tag{11}$$

The Taylor series expansion of $\tau(\tilde{x}_0)$ can be written

$$\tau(x_0+\Delta x_0) = \tau(x_0) + \frac{\partial \tau}{\partial x}(x_0)\Delta x_0 + \text{hot},$$

where $\frac{\partial \tau}{\partial x}(x_0)$ is given by (11). Neglecting the higher order terms, and using the fact that $\tau(x_0) = t$, gives

$$\tau(x_0+\Delta x_0) \approx t + \Delta\tau \tag{12}$$

where

$$\Delta\tau = \frac{-1}{h_t^\top f(x(t))} h_t^\top \Phi(x_0,t)\Delta x_0. \tag{13}$$

The map from initial conditions $\tilde{x}_0 \in U$ to the point where the perturbed trajectory encounters Σ_t can be written as

$$F_t(\tilde{x}_0) = \phi(\tilde{x}_0,\tau(\tilde{x}_0)). \tag{14}$$

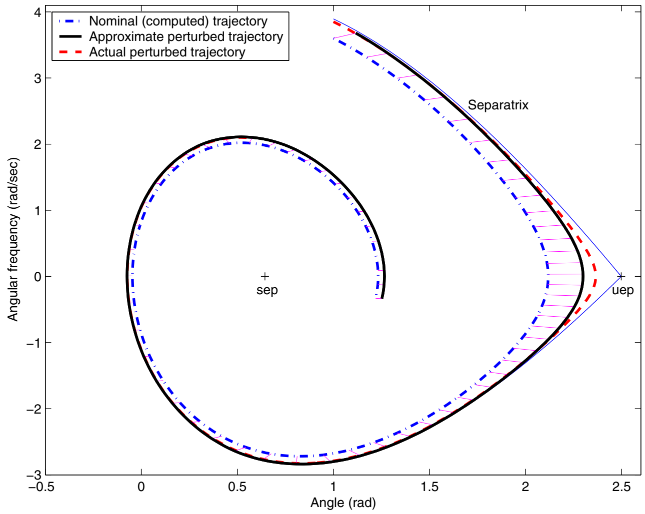

Fig. 5. Phase portrait for highly disturbed case, $x_0^\top = [1\ \ 3.6]$. Hyperplane-based trajectory approximation.

Differentiating gives,

$$\frac{\partial F_t}{\partial x}(x_0) = \Phi(x_0,t) + f(x(t))\frac{\partial \tau}{\partial x}(x_0).$$

Substituting (11) results in,

$$\frac{\partial F_t}{\partial x}(x_0) = \Big(I - \frac{f(x(t))h_t^\top}{h_t^\top f(x(t))}\Big)\Phi(x_0,t). \qquad (15)$$

The Taylor series expansion of $F_t(\tilde{x}_0)$ is,

$$\begin{aligned} F_t(x_0+\Delta x_0) &= \phi\big(x_0+\Delta x_0, \tau(x_0+\Delta x_0)\big) \\ &= F_t(x_0) + \frac{\partial F_t}{\partial x}(x_0)\Delta x_0 + \text{hot}. \end{aligned}$$

Neglecting the higher order terms, and substituting (12) and (15) gives

$$\begin{aligned} &\phi(x_0+\Delta x_0, t+\Delta\tau) \\ &\quad\approx \phi(x_0,t) + \Big(I - \frac{f(x(t))h_t^\top}{h_t^\top f(x(t))}\Big)\Phi(x_0,t)\Delta x_0. \qquad (16) \end{aligned}$$

Equations (13) and (16) fully describe the hyperplane-based approximation to the trajectory emanating from $x_0 + \Delta x_0$.

Substituting (13) into (16) gives

$$\begin{aligned} &\phi(x_0+\Delta x_0, t+\Delta\tau) \\ &\quad\approx \phi(x_0,t) + \Phi(x_0,t)\Delta x_0 + f(x(t))\Delta\tau. \qquad (17) \end{aligned}$$

Comparison with (5) shows that the two forms of trajectory approximation differ only in the contribution due to the time correction $\Delta\tau$. However the following example illustrates that this time correction term can have a substantial effect on the quality of the approximation.

C. Example

Continuing the example of Section II-C, the hyperplane-based approach to trajectory approximation was applied in the more onerous of the two cases, with $x_0^\top = [1\ \ 3.6]$. The time-varying hyperplane was established by defining $h_t \equiv f(t)$, or in other words, the hyperplane was always orthogonal to the nominal trajectory. Results of this approximation process are shown in Figure 5. It can be seen that the approximation underestimates the true perturbed trajectory in the vicinity of the uep, but is otherwise in excellent agreement. A comparison of Figures 4 and 5 shows that this new approach to trajectory approximation gives vastly improved results.

Figure 5 shows lines that separate the nominal and approximate trajectories. Referring to (17), these lines correspond to $(\Phi(x_0,t)\Delta x_0 + f(x(t))\Delta\tau)$, and by definition lie on the time-varying hyperplane Σ_t. The choice of $h_t \equiv f(t)$ implies that these lines should be orthogonal to the nominal trajectory, with

$$\begin{aligned} &f(x(t))^\top\Big(\Phi(x_0,t)\Delta x_0 + f(x(t))\Delta\tau\Big) \\ &\quad= f^\top\Phi(x_0,t)\Delta x_0 + f^\top f\Big(\frac{-1}{f^\top f}f^\top\Phi(x_0,t)\Delta x_0\Big) = 0. \end{aligned}$$

Unfortunately this orthogonality is not apparent in the figure because the horizontal and vertical axes are scaled differently.

IV. Conclusions

The dynamic behaviour of many important large-scale systems is influenced by uncertain parameters. Thorough analysis of the impact of parameter variations is desirable, but may not be computationally feasible. In order to reduce computation, trajectory sensitivities have been used to obtain first-order approximations of perturbed trajectories. These approximations are generally quite accurate, but are detrimentally affected by strong nonlinearities in the vicinity of the stability boundary.

The paper proposes an alternative approach to generating approximate trajectories. Motivated by a state-space view of perturbed trajectories, this new approach utilizes a hyperplane to establish points on the nominal and perturbed trajectories that are adjacent. The perturbation between these adjacent points is then approximated using trajectory sensitivities. This hyperplane-based approach alleviates time skewing between the nominal and perturbed trajectories, and in so doing, provides significantly improved trajectory approximations.

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